# Time asymptotics of $e^{-ith(\kappa)}$ for analytic matrices*

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– *Ratio*: Deinde quod invenis paucis conclusiunculis breviter collige nec modo cures invitationem turbae legentium; paucis ista sat erunt civibus tuis. – *Augustinus*: Ita faciam.

(Aurelius Augustinus, "*Soliloquia*")

**Abstract**

In quantum mechanics the temporal decay of certain resonance states is associated with an effective time evolution $e^{-ith(\kappa)}$, where $h(\cdot)$ is an analytic family of non self-adjoint matrices. In general the corresponding resonance states do not decay exponentially in time. Using analytic perturbation theory, we derive asymptotic expansions for $e^{-ith(\kappa)}$, simultaneously in the limits $\kappa \to 0$ and $t \to \infty$, where the corrections with respect to pure exponential decay have uniform bounds in one complex variable $\kappa^2 t$.
In the Appendix we briefly review analytic perturbation theory, replacing the classical reference to the 1920 book of Knopp [Kn] and its terminology by standard modern references. This might be of independent interest.

## 1 Introduction and results

In this paper we analyze the time evolution $e^{-ith(\kappa)}$ where $h(\kappa)$ satisfies:

**(H1)** Let $\mathscr{H}_0$ be a complex vector space with $\dim\mathscr{H}_0 =: N < \infty$. For $\varepsilon > 0$ define

$$U_\varepsilon(0) := \{z \in \mathbb{C} \,|\, |z| < \varepsilon\}\,, \quad D_\varepsilon(0) := U_\varepsilon(0)\backslash(-\infty, 0]\,.$$

Let $h(\kappa)$, $\kappa \in U_\varepsilon(0)$, denote a family of analytic endomorphisms on $\mathscr{H}_0$ with $h(0) = \lambda_0 \mathbb{1}_{N\times N}$ for some $\lambda_0 \in \mathbb{R}$.

In particular,

$$h(\kappa) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\, h^{(\nu)}(0)\, \kappa^\nu \qquad (\kappa \in U_\varepsilon(0))\,, \tag{1.1}$$

where each $h^{(\nu)}(0)$ is some endomorphism on $\mathscr{H}_0$.

**(H2)** $h'(0) = h^{(1)}(0)$ is self-adjoint.

**(H3)** For $\kappa \in U_\varepsilon(0) \cap \mathbb{R}$ the spectrum $\sigma(h(\kappa))$ of $h(\kappa)$ is contained in the complex closed lower half plane $\overline{\mathbb{C}_-} := \{z \in \mathbb{C} \,|\, \mathrm{Im} z \leq 0\}$.

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We shall derive asymptotic expansions with error bounds which are uniform with respect to $t$ and $\kappa$. In particular, they apply to the simultaneous limit $t \to \infty, \kappa \to 0$. Our interest in this problem comes from quantum mechanics and the time evolution associated with certain resonances which in regular perturbation theory are generated from eigenvalues $\lambda_0$ embedded in the continuous spectrum. More precisely, let $H(\kappa)$ denote an analytic family of semi-bounded self-adjoint operators satisfying the assumptions of [Hu2] (or [KlR]). Then $H(\kappa)$ admits an analytic distortion of abstract Balslev-Combes type, which allows to define resonances as eigenvalues of the deformed operator $H(\kappa, \theta)$, and the embedded eigenvalue $\lambda_0$ of $H(0)$ (with associated spectral projection $\Pi_0$) becomes a discrete eigenvalue of $H(0, \theta)$ for non-real values of $\theta$. If $g \in C_0^\infty$ denotes a cut-off function around $\lambda_0$, the main result of [Hu2] might be stated as

$$\Pi_0 e^{-itH(\kappa)} g(H(\kappa)) \Pi_0 = D(\kappa) e^{-ith(\kappa)} D(\kappa) + \mathcal{R}(\kappa, t)\,, \tag{1.2}$$

where $h(\kappa)$ satisfies assumptions (H1) - (H3) above, with $\mathscr{H}_0 = \mathrm{Ran}(\Pi_0)$, the operators $D(\kappa)$ are explicit and the error term $R(\kappa, t)$ satisfies uniform bounds. We recall that it is well known that $R(\kappa, t)$ necessarily violates any exponential decay estimate as $t \to \infty$ (due to analyticity and semi-boundedness). In [KlR] it has been shown that for $g$ in a Gevrey class of index $a > 1$, $b > 0$ (see [KlR] for the definition of the Gevrey space $\Gamma^{a,b}$) the remainder satisfies (for real $\kappa$)

$$||R(\kappa, t)|| \le O(\kappa^2)\, e^{-Ct^{\frac{1}{a}}} \quad (t \ge 0\,,\ C < ab^{-\frac{1}{a}}\,,\ \kappa \to 0)\,. \tag{1.3}$$

This estimate shows that, for $a$ sufficiently close to 1, the effective time evolution $e^{-ith(\kappa)}$ dominates the remainder term for time scales much larger than the expected physical lifetime $\tau$ determined by the imaginary part of the resonances arising from $\lambda_0$ (generically, in this setting, $\tau$ is of order $\kappa^{-2}$). It is thus natural to investigate the asymptotic behavior of the effective time evolution $e^{-ith(\kappa)}$ as $t \to \infty, \kappa \to 0$. Due to (1.2) and (1.3), this in principle is physically observable. Since $h(\kappa)$ in general is not self-adjoint and since we ask for asymptotics in the *two* parameter family $(\kappa, t)$, this behavior is not trivial, and we do not know any general result in this direction (if $\dim \mathscr{H}_0 \ge 2$).

It is natural to expect that such asymptotics can be obtained by analytic perturbation theory. We shall show that this works. The most obvious correction to pure exponential behavior given by the possibly non-real eigenvalues $\lambda(\kappa)$ of $h(\kappa)$ (the resonances of $H(\kappa)$ originating from $\lambda_0$) are due to possible nilpotent parts $N(\kappa)$ in the Jordan decomposition of $h(\kappa)$, for $\kappa \neq 0$, which give polynomial corrections $O(t^k)$, for some $k \le \dim \mathscr{H}_0$. But there are other sources for possible corrections: Eigenvalues $\lambda(\kappa)$ may branch (i.e., they possess a Puiseux expansion in fractional powers of $\kappa$ around zero), and the associated (individual) eigenprojections $\Pi(\kappa)$ and eigennilpotents $N(\kappa)$ may branch and have poles. We recall that a basic result in analytic perturbation theory (Butler's Theorem) states that branching of eigenvalues implies the existence of poles of eigenprojections. But eigenprojections $\Pi(\kappa)$ and eigennilpotents $N(\kappa)$ may have poles even if no eigenvalue $\lambda(\kappa)$ branches. It turns out that in some sense this is the origin of the most severe corrections to pure exponential behavior, since the order of the associated polynomial correction is in general *not* bounded by $\dim \mathscr{H}_0$. This order may become arbitrarily large, if the convergent Puiseux or power series expansions of some eigenvalues split only in very large order. It is, however, true that in all cases uniform bounds in $(\kappa, t)$ can be reduced to uniform bounds in *one* complex variable, namely in $\kappa^2 t$. We emphasize that this is true in general and does not depend on the stronger hypothesis

**(H4)** All eigenvalues of $h(\kappa)$ have the form $\lambda(\kappa) = \lambda_0 + \alpha\kappa + \beta\kappa^2 + o(|\kappa|^2)$ $(\kappa \to 0)$ with some $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$, where $\mathrm{Im}\beta < 0$.

We recall that hypotheses comparable to (H4) – if $h(\kappa)$ is induced via perturbation theory from a Hamiltonian $H(\kappa)$ – are sometimes denoted as non-vanishing of the Fermi golden rule or as a Fermi golden rule condition; see, e.g., [MS] and [SoW]. Precise asymptotic results for some special cases in this direction are contained in Section 3, while Theorem 1.5 below (which holds more generally) only gives upper bounds (in equation (1.28) and (1.29)). In particular the norm estimate in Corollary 1.6 shows that the results in [KlRWü] are optimal for the case considered in [KlRWü]. We recall that there $N = \dim \mathscr{H}_0$ is not necessarily finite (and thus, in particular, nothing can be assumed on the Jordan decomposition). Furthermore, (H4) is assumed to hold, thus the physical lifetime is $O(\kappa^{-2})$ and $O(t\kappa^2)$ is *not* negligible compared to 1 if the time $t$ is of the order of the physical lifetime. To obtain valid estimates up to and possibly beyond the physical lifetime one simultaneously needs results comparable to [KlR] and, for the finite dimensional part, the results of this paper.

We emphasize: Although individual projections and eigennilpotents may have poles, the effective time evolution $e^{-ith(\kappa)}$ is analytic in both variables $(\kappa, t)$. Thus there must be substantial cancelation in the sum over all generalized eigenspaces. The problem is to estimate this sum, uniformly in $(\kappa, t)$. To describe our result in more detail we need some preliminary considerations on the reduction of $h(\kappa)$ to spectral subspaces associated with certain clusters of eigenvalues.

Assume (H1). Then a basic result in analytic perturbation theory (Theorem 2.3 in [K2, II §2.3]) implies that each eigenvalue $\lambda(\cdot)$ is of the form

$$\lambda(\kappa) = \lambda_0 + \alpha\kappa + O(|\kappa|^{1+1/n}) \qquad (\kappa \to 0) \tag{1.4}$$

for some $n \leq N$ and some $\alpha \in \mathbb{C}$. Assuming in addition (H2), $\alpha$ is real, but this is not needed yet. We now partition all eigenvalues with the same value of $\alpha$ (say $\alpha = \alpha_k$) into one cluster $\mathcal{C}_k$, $1 \leq k \leq R \leq N = \dim \mathscr{H}_0$. We call $\mathcal{C}_k$ maximal in linear order. Since the diameter of $\mathcal{C}_k$ is of order $o(|\kappa|)$, $\kappa \to 0$,

$$\operatorname{dist}(\mathcal{C}_k, \mathcal{C}_{k'}) \geq c|\kappa| \tag{1.5}$$

for all $k' \neq k$ and some $c > 0$. Thus there exists a contour $\Gamma_{\mathcal{C}_k}(\cdot)$, separating $\mathcal{C}_k$ from the rest of the spectrum $\sigma(h(\cdot))$, such that

$$c'|\kappa| \geq \operatorname{dist}(z, \sigma(h(\kappa))) \geq c|\kappa| \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)\,,\ \kappa \in U_\varepsilon(0))\,, \tag{1.6}$$

where $c', c$ are some strictly positive constants and $\varepsilon > 0$ is taken sufficiently small. The corresponding maximal cluster projection is given by the Riesz projection

$$\Pi_{\mathcal{C}_k}(\kappa) := -\frac{1}{2\pi i} \oint_{\Gamma_{\mathcal{C}_k}(\kappa)} (h(\kappa) - z)^{-1}\, dz\,, \tag{1.7}$$

where we can choose $\Gamma_{\mathcal{C}_k}(\kappa)$ to be a counterclockwise oriented circle with center

$$J^{(\mathcal{C}_k)}(\kappa) := \lambda_0 + \alpha_k \kappa \tag{1.8}$$

and radius $\rho_k(\kappa) = c_k|\kappa|$, for some $c_k > 0$.

**Proposition 1.1** Assume (H1). Let $\mathcal{C}_k$ be a cluster, which is maximal in linear order. Then the corresponding maximal cluster projection $\Pi_{\mathcal{C}_k}(\cdot)$, defined in (1.7), is analytic in $U_\varepsilon(0)$ for $\varepsilon > 0$ sufficiently small.

This is a basic result and is implicitly contained in [K2]. For the sake of the reader, we shall include a formal proof in Section 2.1. We further recall that any analytic family of projections $\Pi_{\mathcal{C}_k}(\kappa)$ reducing $h(\kappa)$ can be simultaneously transformed to the family $\Pi_{\mathcal{C}_k}(0)$ with fixed $\kappa = 0$.[1] More precisely, there exists an operator-valued function

$$\begin{aligned} \mathcal{U}(\cdot):\ U_\varepsilon(0) &\to B(\mathscr{H}_0) \\ \kappa &\mapsto \mathcal{U}(\kappa) \end{aligned}$$

such that (cf. [K2, II §4.2 and §4.5]) $\mathcal{U}(0) = \mathbb{1}$ and

[1]This fact was discovered by Kato [K1] in context with the adiabatic theorem in quantum mechanics. We call it *adiabatic reduction*.

i) $\mathcal{U}(\kappa)^{-1}$ exists for all $\kappa \in U_\varepsilon(0)$ and $\mathcal{U}(\cdot)^{\pm 1}$ are analytic in $U_\varepsilon(0)$,

ii) $\mathcal{U}(\cdot)$ simultaneously transforms all maximal cluster projections $\Pi_{\mathcal{C}_k}(\cdot)$, i.e.:

$$\mathcal{U}(\kappa)\Pi_{\mathcal{C}_k}(0)\mathcal{U}(\kappa)^{-1} = \Pi_{\mathcal{C}_k}(\kappa) \qquad (k \in \{1, \dots, R\}\,,\ \kappa \in U_\varepsilon(0)). \tag{1.9}$$

$\mathcal{U}(\cdot)$ is called a simultaneous transformation function of $\Pi_{\mathcal{C}_k}(\cdot)$; cf. [K2, II §4.5 and §4.2]. A straightforward calculation shows that

$$\widetilde{h}(\kappa) := \mathcal{U}(\kappa)^{-1} h(\kappa)\mathcal{U}(\kappa) \qquad (\kappa \in U_\varepsilon(0)) \tag{1.10}$$

is reduced by the family $\Pi_{\mathcal{C}_k}(0)$, i.e.,

$$h(\kappa)\Pi_{\mathcal{C}_k}(\kappa) = \mathcal{U}(\kappa)\widetilde{h}(\kappa)\Pi_{\mathcal{C}_k}(0)\mathcal{U}(\kappa)^{-1} \tag{1.11}$$

for all $\kappa \in U_\varepsilon(0)$ and $k \in \{1, \dots, R\}$. In particular,

$$\sigma(h(\kappa)) = \sigma(\widetilde{h}(\kappa)) \qquad (\kappa \in U_\varepsilon(0))\,.$$

Using (1.11), estimates on $e^{-ith(\kappa)}$ reduce to estimates on the single "blocks" $\widetilde{h}(\kappa)\Pi_{\mathcal{C}_k}(0)$. Thus, without loss of generality, we are reduced to the model situation

**(WLG 1)** There is an $\alpha \in \mathbb{C}$ such that all $\lambda(\cdot) \in \sigma(h(\cdot))$ have the form

$$\lambda(\kappa) = \lambda_0 + \alpha\kappa + O(|\kappa|^{1+\frac{1}{N}}) \qquad (\kappa \to 0)\,. \tag{1.12}$$

**Lemma 1.2** Assume (H1), (H2) and (WLG1). Then $\alpha$ appearing in (1.12) is real and $h'(0) = \alpha\,\mathbb{1}$.

**Proof**: See Section 2.1.

Assuming (H1), (H2) and (WLG1), we obtain as before (using Theorem 2.3 in [K2, II §2.3]) that each eigenvalue $\lambda(\cdot)$ is of the form

$$\lambda(\kappa) = \lambda_0 + \alpha\kappa + \beta\kappa^2 + O(|\kappa|^{2+1/n}) \qquad (\kappa \to 0) \tag{1.13}$$

for some $n \le N$ and some $\beta \in \mathbb{C}$. We now partition all eigenvalues with the same value of $\beta$ (say $\beta = \beta_k$) into one cluster $\mathcal{C}_k$, $1 \le k \le R \le N = \dim\mathscr{H}_0$. We call $\mathcal{C}_k$ maximal in quadratic order. Since the diameter of $\mathcal{C}_k$ is of order $o(|\kappa|^2)$, $\kappa \to 0$, one has

$$\mathrm{dist}(\mathcal{C}_k, \mathcal{C}_{k'}) \ge c|\kappa|^2 \tag{1.14}$$

for all $k' \ne k$ and some $c > 0$. Thus there exists a contour $\Gamma_{\mathcal{C}_k}(\cdot)$, separating $\mathcal{C}_k$ from the rest of the spectrum $\sigma(h(\cdot))$, such that

$$c'|\kappa|^2 \ge \mathrm{dist}(z, \sigma(h(\kappa))) \ge c|\kappa|^2 \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)\,,\ \kappa \in U_\varepsilon(0))\,, \tag{1.15}$$

where $c', c$ are some strictly positive constants and $\varepsilon > 0$ is sufficiently small. The corresponding maximal cluster projection is given by the Riesz projection

$$\Pi_{\mathcal{C}_k}(\kappa) := -\frac{1}{2\pi i} \oint_{\Gamma_{\mathcal{C}_k}(\kappa)} (h(\kappa) - z)^{-1}\, dz\,, \tag{1.16}$$

where we can choose $\Gamma_{\mathcal{C}_k}(\kappa)$ to be a counterclockwise oriented circle with center

$$J^{(\mathcal{C}_k)}(\kappa) := \lambda_0 + \alpha\kappa + \beta_k\kappa^2 \tag{1.17}$$

and radius $\rho_k(\kappa) = c_k\,|\kappa|^2$ for some $c_k > 0$. In this situation, i.e., under stronger assumptions, one has in analogy to Proposition 1.1

**Proposition 1.3** Assume (H1), (H2) and (WLG1). Let $\mathcal{C}_k$ be a cluster, which is maximal in quadratic order. Then the corresponding maximal cluster projection $\Pi_{\mathcal{C}_k}(\cdot)$, defined in (1.16), is analytic in $U_\varepsilon(0)$ for $\varepsilon > 0$ sufficiently small. Furthermore, $\Pi_{\mathcal{C}_k}(0)$ is self-adjoint.

**Proof**: See Section 2.1.

As above, we now may simultaneously transform the projections $\Pi_{\mathcal{C}_k}(\kappa)$ (where the cluster $\mathcal{C}_k$ is maximal in quadratic order) to $\Pi_{\mathcal{C}_k}(0)$ via an analytic map $\mathcal{U} : U_\varepsilon(0) \to B(\mathscr{H}_0)$, and set

$$\widetilde{h}(\kappa) = \mathcal{U}(\kappa)^{-1} h(\kappa) \mathcal{U}(\kappa). \tag{1.18}$$

Using Proposition 1.3, each block $\widetilde{h}(\kappa)\Pi_{\mathcal{C}_k}(0)$, considered as an operator on $\operatorname{Ran}\Pi_{\mathcal{C}_k}(0)$, still satisfies (H1), (H2) and (WLG1). By Lemma 1.2, this shows that we can assume without loss of generality:

**(WLG 2)** There exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ such that $h'(0) = \alpha \mathbb{1}$ and all $\lambda(\cdot) \in \sigma(h(\cdot))$ have the form

$$\lambda(\kappa) = J(\kappa) + O(|\kappa|^{2+\frac{1}{N}}) \quad (\kappa \to 0)\,, \qquad J(\kappa) := \lambda_0 + \alpha\kappa + \beta\kappa^2\,. \tag{1.19}$$

In particular,

**Corollary 1.4** Assume (H1), (H2) and (WLG2). Then, for all $\kappa \in U_\varepsilon(0)$,

$$h(\kappa) - J(\kappa) = O(|\kappa|^2) \quad (\kappa \to 0)\,. \tag{1.20}$$

The following theorem (Theorem 1.5) is our main result. It describes the time asymptotics of $e^{-ith(\kappa)}$ in the limit $\kappa \to 0$, uniformly for $t \in \mathbb{C}$. Theorem 1.5 and its proof need the functions

$$\exp_n(z) := \sum_{\nu=n}^{\infty} \frac{z^\nu}{\nu!}\,, \qquad \widetilde{\exp}_n(z) := z^{-n} \exp_n(z) \qquad (z \in \mathbb{C}\,,\ n \in \mathbb{N}) \tag{1.21}$$

and the estimate

$$|\widetilde{\exp}_n(z)| \le \frac{1}{n!}\left(1 + |z| + \frac{|z|^2}{2} + \frac{|z|^3}{2\cdot 3} + \dots\right) = \frac{1}{n!} e^{|z|} \qquad (n \in \mathbb{N}\,,\ z \in \mathbb{C})\,. \tag{1.22}$$

Now let $\lambda_\mu(\cdot)$ denote the pairwise distinct eigenvalues of $h(\cdot)$. Let $\Pi_\mu(\cdot)$ denote the (possibly singular at $\kappa = 0$) eigenprojection associated with the eigenvalue $\lambda_\mu(\cdot)$ and $N_\mu(\cdot)$ the corresponding eigennilpotent. Using the Jordan decomposition $h(\kappa) = \sum_\mu \lambda_\mu(\kappa)\Pi_\mu(\kappa) + N_\mu(\kappa)$ one has

$$e^{-ith(\kappa)} = \sum_\mu \Pi_\mu(\kappa) e^{-ith(\kappa)} = \sum_\mu \Pi_\mu(\kappa) e^{-it\lambda_\mu(\kappa)} e^{-itN_\mu(\kappa)}\,. \tag{1.23}$$

We shall estimate (1.23) for $\kappa \to 0$, uniformly in $t$, using the properties of the eigenvalues $\lambda_\mu(\cdot)$ and the structure of the Jordan decomposition (in particular the order of the possible poles in $\Pi_\mu(\cdot)$ and $N_\mu(\cdot)$):

**Theorem 1.5** Assume (H1), (H2) and (WLG 2). Define

$$J_\mu(\cdot) := \lambda_\mu(\cdot) - J(\cdot) \tag{1.24}$$

for $\mu \in \{1, \ldots, S\}$ for some $1 \leq S \leq N$. Let $\widetilde{N} \in \mathbb{N}\backslash\{0\}$. Then

$$e^{-ith(\kappa)} = e^{-itJ(\kappa)}(\mathbb{1} + F_{\widetilde{N}}(\kappa, t) + R_{\widetilde{N}}(\kappa, t)) \qquad (\kappa \in U_\varepsilon(0),\ t \in \mathbb{C}), \tag{1.25}$$

where

$$\begin{aligned} F_{\widetilde{N}}(\kappa, t) \quad &:= -\mathbb{1} + \sum_{\mu=1}^{S} \Big( \sum_{k=0}^{\widetilde{N}-1} \tfrac{1}{k!} \left(-itJ_\mu(\kappa)\right)^k \Big) \Pi_\mu(\kappa)\, e^{-itN_\mu(\kappa)} \\ &=: \sum_{k=1}^{\widetilde{N}+N-2} \tfrac{1}{k!} \left(-it\right)^k F_k(\kappa)\,, \end{aligned} \tag{1.26}$$

and

$$R_{\widetilde{N}}(\kappa, t) := \sum_{\mu=1}^{S} \exp_{\widetilde{N}}(-itJ_\mu(\kappa))\Pi_\mu(\kappa) e^{-itN_\mu(\kappa)}\,. \tag{1.27}$$

The functions $F_{\widetilde{N}}(\cdot, \cdot)$ and $R_{\widetilde{N}}(\cdot, \cdot)$ are analytic in both variables $\kappa \in U_\varepsilon(0)$ and $t \in \mathbb{C}$. Let $p \geq 2 + \frac{1}{N}$ be the number given in Remark 2.3.[2] Choosing $\widetilde{N} \geq (p-2)(N-1)N$, then there exists a constant $c > 0$ such that for all $\kappa \in U_\varepsilon(0)$ the remainders $F_{\widetilde{N}}(\cdot, \cdot)$ and $R_{\widetilde{N}}(\cdot, \cdot)$ satisfy

$$\|F_{\widetilde{N}}(\kappa, t)\| \leq \sum_{k=1}^{\widetilde{N}+N-2} O(|t\kappa^2|^k)\,, \tag{1.28}$$

$$\|R_{\widetilde{N}}(\kappa, t)\| \leq e^{c\,|t|\,|\kappa|^{2+\frac{1}{N}}} \sum_{k=0}^{N-1} O(|t\kappa^2|^{k+\widetilde{N}}) \tag{1.29}$$

as $\kappa \to 0$, uniformly for $t \in \mathbb{C}$.

**Proof**: See Section 2.2.

The results given in Theorem 1.5 immediately lead to

**Corollary 1.6** Assume (H1), (H2), (WLG 2), (H3), (H4) and the same notation as in Theorem 1.5. Let $\widetilde{N} \geq (p-2)(N-1)N$. Then, for $\kappa \in U_\varepsilon(0) \cap \mathbb{R}$ and $t \geq 0$,

$$\begin{aligned} \|e^{-ith(\kappa)}\| = e^{-t|\kappa|^2|\mathrm{Im}\,\beta|} \Big(1 + &\sum_{k=1}^{N+\widetilde{N}-2} O(|t\kappa^2|^k) \\ &+ e^{ct\,|\kappa|^{2+\frac{1}{N}}} \sum_{k=0}^{N-1} O(|t\kappa^2|^{k+\widetilde{N}})\Big) \end{aligned} \tag{1.30}$$

as $\kappa \to 0$, uniformly for $t \geq 0$. ($c$ is the strictly positive constant found in Theorem 1.5.)

We remark that some of the results of this paper are contained in the thesis [R], in a somewhat preliminary form. Our present partition of eigenvalues into different clusters has some similarity to the presentation of degenerate asymptotic perturbation theory in [HuP] and [Hu1].

The outline of the paper is as follows: In Section 2 we prove the statements in this introduction. In Section 3 we sharpen Theorem 1.5 in some special cases of eigenvalue splitting

[2] $p$ is the smallest order in which all non permanent degenerations of eigenvalue branches are lifted. It is in understanding $p$ that the full machinery of analytic perturbation theory is required.

to exhibit the leading term in the remainder terms $F_{\widetilde{N}}$ and $R_{\widetilde{N}}$. This gives lower bounds on the remainder. In Appendix A we give a short summary on algebraic functions, and in Appendix B we recall the structure of the Jordan decomposition in analytic perturbation theory. All the results in the appendices are well known. We have two reasons to include the appendices in this paper:

First, the standard exposition of algebraic functions in analytic perturbation theory in [K2] is based on the 1920 book of Knopp [Kn]. [3] Reed and Simon [ReSi] refer to [K2] and the English translation of Knopp's book. Baumgärtel's book [B] is self contained, but, strictly speaking, its notion of a Riemann surface is in some sense closer to the thinking of Weierstraß than to the modern definition.[4] In our opinion, this state of the literature hardly is an adequate presentation of analytic perturbation theory, which after all is a central part of modern mathematical physics. Thus we think that it might be in the interest of the reader to find a short summary on algebraic functions in Appendix A and on unique factorization[5] of polynomials with analytic coefficients in Appendix B which is exclusively based on standard modern references.[6]

Second, a consistent formulation of the structure of the Jordan decomposition (in Appendix B) is handy for the proof of our results in Section 3.

# 2 Proofs

## 2.1 Preliminaries

To prove Proposition 1.1 we use the following lemma, taken from [K2, I §4.2, (4.12)]:

**Lemma 2.1** Let $V$ be a finite-dimensional linear space with $\dim V := N < \infty$. Let $T \in \mathrm{GL}(V)$, where $\mathrm{GL}(\cdot)$ denotes the general linear group. Then

$$\|T^{-1}\| \leq C \, \frac{\|T\|^{N-1}}{|\det T|} \,,$$

where $C$ is a constant independent of $T$ but depending on the norm employed. If $V$ is a Hilbert space, one can set $C = 1$.

**Proof of Proposition 1.1**: We claim that for $z \in \Gamma_{\mathcal{C}_k}(\kappa)$ one gets

$$h(\kappa) - z = O(|\kappa|) \qquad (\kappa \to 0)\,. \tag{2.1}$$

Indeed, using (1.7), each $z \in \Gamma_{\mathcal{C}_k}(\kappa)$ can be written as

$$z = J^{(\mathcal{C}_k)}(\kappa) + \rho_k(\kappa) e^{i\varphi}\,, \qquad \rho_k(\kappa) = c_k |\kappa| \tag{2.2}$$

for some $c_k > 0$, some $\varphi \in [0, 2\pi)$ and all $\kappa \in U_\varepsilon(0)$. Combining (2.2) and (1.8) yields

$$z = \lambda_0 + O(|\kappa|) \qquad (\kappa \to 0,\ z \in \Gamma_{\mathcal{C}_k}(\kappa))\,. \tag{2.3}$$

[3] Strictly speaking, the theory of algebraic functions in [Kn] does not cover the situation considered in Kato's book [K2], and this is acknowledged in the footnote on p.64 in [K2].

[4] In spirit, the presentation of algebraic functions is an amplified version of [Kn], removing the above mentioned restrictions in [Kn]. The resulting class of functions is called *algebroidal*. One of the main differences to a modern presentation is in omitting the construction of a Riemann surface as a topological Hausdorff space, starting from the local data (see, e.g., [Fo, 6.7], "The Topological Space Associated to a Presheaf").

[5] This factorization is neither treated in [Kn] nor in [K2] which refers the reader to [B].

[6] These subjects concern function theory on Riemann surfaces and some basic material from algebraic geometry. Since the publication of [B], a number of good textbooks in these fields were published which at least for the beginning student have considerably transformed the list of natural references (e.g., [Fo], [BrKnö], [GH]).

Then combining (2.3) and $h(\kappa) = \lambda_0 \mathbb{1} + h'(0)\kappa + O(|\kappa|^2)$ $(\kappa \to 0)$ proves (2.1).
Lemma 2.1 with $C = 1$ gives

$$\|(h(\kappa) - z)^{-1}\| \leq \frac{\|h(\kappa) - z\|^{N-1}}{|\det(h(\kappa) - z)|} \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)) \,. \tag{2.4}$$

To estimate the denominator in (2.4) we use Cramer's rule and (1.6) to get

$$|\det(h(\kappa) - z)| = \prod_{k'} \prod_{\lambda_\nu \in \mathcal{C}_{k'}} |\lambda_\nu(\kappa) - z|^{m(\lambda_\nu)} \geq c\,|\kappa|^N \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)) \,, \tag{2.5}$$

where $m(\lambda_\nu)$ denotes the algebraic multiplicity of the eigenvalue $\lambda_\nu(\kappa)$ and $c$ is some strictly positive constant. Thus combining (2.5), (2.4) and (2.1) gives

$$\|(h(\kappa) - z)^{-1}\| = O(|\kappa|^{-1}) \qquad (\kappa \to 0\,,\ z \in \Gamma_{\mathcal{C}_k}(\kappa)) \,. \tag{2.6}$$

Then combining (2.6), (2.2) and (1.7) yields

$$\|\Pi_{\mathcal{C}_k}(\kappa)\| \leq \rho_k(\kappa) \max_{z \in \Gamma_{\mathcal{C}_k}(\kappa)} \|(h(\kappa) - z)^{-1}\| = O(1) \qquad (\kappa \to 0) \,. \tag{2.7}$$

It is clear from (1.7) that $\Pi_{\mathcal{C}_k}(\cdot)$ is analytic in the punctured disc $U_\varepsilon(0)\backslash\{0\}$. By (2.7) the isolated singularity in zero is removable. Thus $\Pi_{\mathcal{C}_k}(\cdot)$ is analytic in $U_\varepsilon(0)$ by Riemann's Removable Singularity Theorem.

■

**Proof of Lemma 1.2**: By the reduction process in [K2, II §2.3, (2.40)], the eigenvalues of $h'(0)$ coincide with the coefficient $\alpha$ in the expansion (1.12) of $\lambda(\kappa) \in \sigma(h(\kappa))$. Thus, by (WLG1) and (H2), $\alpha \in \mathbb{R}$ is the only eigenvalue of the self-adjoint operator $h'(0)$. By the spectral theorem there is a unitary transformation $U$ with $h'(0) = U(\alpha \mathbb{1})U^{-1} = \alpha \mathbb{1}$.

■

**Proof of Proposition 1.3**: By Lemma 1.2 and the definition of $\Gamma_{\mathcal{C}_k}(\kappa)$ in (1.16) one has

$$h(\kappa) - z \quad = \quad O(|\kappa|^2) \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)\,,\ \kappa \to 0) \,.$$

Furthermore, by (1.12) and the definition of $\Gamma_{\mathcal{C}_k}(\kappa)$, any eigenvalue $\lambda(\kappa)$ of $h(\kappa)$ satisfies

$$|\lambda(\kappa) - z| \geq c|\kappa|^2 \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)) \tag{2.8}$$

for some $c > 0$. Thus, for some $c > 0$,

$$|\det(h(\kappa) - z)| \geq c|\kappa|^{2N} \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)) \,. \tag{2.9}$$

Arguing as in the proof of Proposition 1.1, the estimates (2.8) and (2.9) lead to

$$\|(h(\kappa) - z)^{-1}\| = O(|\kappa|^{-2}) \qquad (z \in \Gamma_{\mathcal{C}_k}(\kappa)\,,\ \kappa \to 0) \tag{2.10}$$

and thus finally to

$$\|\Pi_{\mathcal{C}_k}(\kappa)\| = O(1) \qquad (\kappa \to 0) \,. \tag{2.11}$$

Thus $\Pi_{\mathcal{C}_k}(\cdot)$ is analytic in $U_\varepsilon(0)$ by Riemann's removable singularity Theorem.

To show the self-adjointness of $\Pi_{\mathcal{C}_k}(0)$, observe that a short calculation gives

$$\Pi^*_{\mathcal{C}_k}(\kappa) = -\frac{1}{2\pi i} \oint_{\Gamma_{\mathcal{C}_k}(\kappa)} (h^*(\kappa) - z)^{-1}\, dz \qquad (\kappa \in U_\varepsilon(0)) \,.$$

Thus, by the second resolvent equation,

$$\begin{aligned} \Pi_{\mathcal{C}_k}(\kappa) - \Pi^*_{\mathcal{C}_k}(\kappa) &= -\frac{1}{2\pi i} \oint_{\Gamma_{\mathcal{C}_k}(\kappa)} (h(\kappa) - z)^{-1}\big(h^*(\kappa) - h(\kappa)\big)(h^*(\kappa) - z)^{-1}\, dz \\ &= O(|\kappa|) \qquad (\kappa \to 0) \,, \end{aligned}$$

where we have used (H2), (2.2) and (2.6) for $h(\kappa)$ and $h^*(\kappa)$. Taking the limit $\kappa \to 0$ gives the self-adjointness of $\Pi_{\mathcal{C}_k}(0)$.

■

**Proposition 2.2** Assume (H1), (H2), (WLG2) and the notation of Theorem 1.5. Then there exists $p \in [2 + \frac{1}{N}, \infty)$ such that all eigenprojections satisfy the (possibly very rough) estimate

$$\Pi_\mu(\kappa) = O(|\kappa|^{(-p+2)(N-1)}) \qquad (\kappa \to 0\,,\ \mu \in \{1, \ldots, S\})\,. \tag{2.12}$$

As indicated below in Remark 2.3, the number $p$ is in principle explicit.

**Remark 2.3** Assuming (H1), (H2) and (WLG2), we denote (exactly as in Theorem 1.5) the pairwise distinct eigenvalues of $h(\cdot)$ by $\lambda_\mu(\cdot)$ ($\mu \in \{1, \ldots, S\}$) for some $1 \leq S \leq N$. Then we can uniquely identify each index $\mu$ with a pair $(\ell, j)$ (where $j = j(\ell)$), introduced in Appendix A.2. That is, $\lambda_\mu(\cdot)$ belongs to a cycle (the $\ell$th) with period $n_\ell$ and branches in generation $k_\ell \leq \infty$ (cf. Appendix A.2 and Definition B.10). The assumption (WLG2) implies $k_\ell \geq 2$. Note that $1 \leq n_\ell \leq N$. By (B.15) and (WLG2) all $\lambda_\mu(\cdot)$ are of the form

$$\begin{aligned}
\lambda_\mu(\kappa) = \lambda_{\ell,j}(\kappa) &= \sum_{\nu=0}^{\infty} a_\nu^{(\ell,j)} \kappa^{\frac{\nu}{n_\ell}} \\
&= J(\kappa) + \sum_{\nu=1}^{\infty} a_{2n_\ell+\nu}^{(\ell,j)} \kappa^{2+\frac{\nu}{n_\ell}} && (2.13)\\
&= J(\kappa) + (J_\ell^{(k_\ell)}(\kappa) - J(\kappa)) + J_{\ell,j}^{(k_\ell,\infty)}(\kappa)\,, && (2.14)
\end{aligned}$$

where $J_\ell^{(k_\ell)}(\cdot)$ and $J_{\ell,j}^{(k_\ell,\infty)}(\cdot)$ are defined in (B.25) and (B.26). In (2.14), the term $J_\ell^{(k_\ell)}(\kappa) - J(\kappa)$ is zero, if $k_\ell = 2$. If $\sigma(h(\cdot))$ contains one or none 1-cycle, then it contains at least one cycle with branching generation $k_\ell < \infty$ and $p$ is determined by the biggest branching generation $\neq \infty$:

$$p := \max_\ell \{k_\ell + \frac{1}{n_\ell}\,;\ k_\ell < \infty\} < \infty\,. \tag{2.15}$$

If $\sigma(h(\cdot))$ contains more than one 1-cycle, then there exists a lowest order $K_* \in \mathbb{N}$ in which none of the power series expansions of these 1-cycles coincide. (Otherwise the eigenvalues given by these 1-cycles would not be pairwise distinct.) (WLG2) implies $K_* \geq 3$. Thus, if $\sigma(h(\cdot))$ contains more than one 1-cycle, then $p$ is given by

$$p := \max_{\ell;\ k_\ell < \infty} \{k_\ell + \frac{1}{n_\ell}\,,\ K_*\} < \infty\,. \tag{2.16}$$

If $\sigma(h(\cdot))$ consists of just one 1-cycle, then the corresponding eigenprojection is equal to $\mathbb{1}$. In this case Proposition 2.2 is pointless.

**Proof of Proposition 2.2**: From (2.13) combined with (WLG2) it follows that

$$|\lambda_\mu(\kappa) - \lambda_{\mu'}(\kappa)| = |\kappa|^2 \left| \sum_{\nu=1}^{\infty} a_{n_\mu+\nu}^{(\mu)} \kappa^{\frac{\nu}{n_\mu}} - \sum_{\nu=1}^{\infty} a_{n_{\mu'}+\nu}^{(\mu')} \kappa^{\frac{\nu}{n_{\mu'}}} \right| \tag{2.17}$$

for all $\mu, \mu' \in \{1, \ldots, S\}$. Obviously, this implies

$$|\lambda_\mu(\kappa) - \lambda_{\mu'}(\kappa)| \leq c|\kappa|^{2+\frac{1}{N}} \tag{2.18}$$

for some $c > 0$. Furthermore, by (2.17), there exist $p \in [2 + \frac{1}{N}, \infty)$ and $c > 0$ such that for all $\mu \neq \mu$

$$|\lambda_\mu(\kappa) - \lambda_{\mu'}(\kappa)| \geq c|\kappa|^p \,. \tag{2.19}$$

Estimate (2.19) gives that for each $\lambda_\mu(\cdot)$ there exists a contour $\Gamma_\mu(\cdot)$ in the complex plane which separates $\lambda_\mu(\cdot)$ from the rest of $\sigma(h(\cdot))$. In particular, by use of (2.19), one can choose $\Gamma_\mu(\kappa)$ to be a counterclockwise oriented circle with radius $\varrho|\kappa|^p$ (for some $c > \varrho > 0$) and center $\lambda_\mu(\kappa)$. Then any $z \in \Gamma_\mu(\kappa)$ can be written as

$$z = \lambda_\mu(\kappa) + \varrho|\kappa|^p e^{i\varphi} \tag{2.20}$$

for some $\varrho > 0$ and some $\varphi \in [0, 2\pi)$, and

$$\Pi_\mu(\kappa) := -\frac{1}{2\pi i} \oint\limits_{\Gamma_\mu(\kappa)} (h(\kappa) - z)^{-1} \, dz \,. \tag{2.21}$$

Obviously,

$$\|\Pi_\mu(\kappa)\| \leq \varrho|\kappa|^p \sup_{z \in \Gamma_\mu(\kappa)} \|(h(\kappa) - z)^{-1}\| \,. \tag{2.22}$$

Applying Lemma 2.1 with $C = 1$ to $h(\kappa) - z$ for $z \in \Gamma_\mu(\kappa)$ yields

$$\|(h(\kappa) - z)^{-1}\| \leq \frac{\|h(\kappa) - z\|^{N-1}}{|\det(h(\kappa) - z)|} = \frac{\|h(\kappa) - z\|^{N-1}}{\prod\limits_{\mu'=1}^{S} |\lambda_{\mu'}(\kappa) - z|^{m(\lambda_{\mu'})}} \,, \tag{2.23}$$

where $m(\lambda_{\mu'})$ denotes the algebraic multiplicity of $\lambda_{\mu'}(\kappa)$. (Recall that $1 \leq m(\lambda_{\mu'}) \leq N$.) To estimate the denominator on the r.h.s in (2.23) we observe that, as $\kappa \to 0$,

$$\begin{aligned} |\lambda_{\mu'}(\kappa) - z| &\overset{(2.20)}{=} \left|\lambda_{\mu'}(\kappa) - \lambda_\mu(\kappa) - \varrho|\kappa|^p e^{i\varphi}\right| \overset{(2.19)}{\geq} \left|c|\kappa|^p - \varrho|\kappa|^p\right| = |c - \varrho| \; |\kappa|^p \\ &=: \; c'|\kappa|^p \,, \end{aligned} \tag{2.24}$$

where $c' > 0$. Combining (2.20), (1.24) and (1.20) gives

$$h(\kappa) - z = O(|\kappa|^2) \qquad (\kappa \to 0 \,, z \in \Gamma_\mu(\kappa)) \,. \tag{2.25}$$

Then inserting (2.25) and (2.24) into (2.23) gives

$$\|(h(\kappa) - z)^{-1}\| \leq \frac{O(|\kappa|^{2(N-1)})}{|\kappa|^{pN}} = O(|\kappa|^{2(N-1)-pN}) \qquad (\kappa \to 0 \,, z \in \Gamma_\mu(\kappa)) \,. \tag{2.26}$$

Finally, using (2.26) in (2.22) proves (2.12).

■

**Corollary 2.4** Assume (H1), (H2), (WLG2) and the notation introduced in Theorem 1.5. Let $p \in [2 + \frac{1}{N}, \infty)$ be the number from Proposition 2.2. Then, for all $\mu \in \{1, \ldots, S\}$,

$$J_\mu(\kappa) = O(|\kappa|^{2+\frac{1}{N}}) \,, \tag{2.27}$$

$$J_\mu(\kappa)^n N_\mu(\kappa)^m = O(|\kappa|^{2m}) J_\mu(\kappa)^n \Pi_\mu(\kappa) \qquad (m \in \mathbb{N}\backslash\{0\}, \; n \in \mathbb{N}) \,, \tag{2.28}$$

$$J_\mu(\kappa)^n \Pi_\mu(\kappa) = O(|\kappa|^{(-p+2)(N-1)+(2+\frac{1}{N})n}) \qquad (n \in \mathbb{N}) \,, \tag{2.29}$$

as $\kappa \to 0$. In particular, for all $n \geq (2 + \frac{1}{N})^{-1} (p-2)(N-1)$,

$$J_\mu(\kappa)^n \Pi_\mu(\kappa) = O(1) \qquad (\kappa \to 0) \,. \tag{2.30}$$

If $p = 2 + \frac{1}{N}$, then one gets the better estimate

$$J_\mu(\kappa)^n \Pi_\mu(\kappa) = O(1) \qquad (\kappa \to 0, \; n \geq 1) \,. \tag{2.31}$$

**Proof**: (1.24), (2.13) and $n_\ell \le N$ prove (2.27). Estimate (2.29) is proven by combining (2.27) and (2.12). Since $\Pi_\mu(\cdot)$ is a projection and $h(\cdot)$ and $\Pi_\mu(\cdot)$ commute, one has

$$N_\mu(\kappa)^m = \big((h(\kappa)-\lambda_\mu(\kappa))\,\Pi_\mu(\kappa)\big)^m \overset{(1.24)}{=} (h(\kappa)-J(\kappa)-J_\mu(\kappa))^m\Pi_\mu(\kappa) \tag{2.32}$$

$$\overset{(1.20)}{=} O(|\kappa|^{2m})\Pi_\mu(\kappa) \qquad (\kappa\to 0\,,\ m\in\mathbb{N}\backslash\{0\})\,. \tag{2.33}$$

This proves (2.28).

Observing that

$$(2+\frac{1}{N})n + (-p+2)(N-1) \ge 0$$

is equivalent to

$$n \ge (2+\frac{1}{N})^{-1}\,(p-2)(N-1) \tag{2.34}$$

proves (2.30).

Estimate (2.31) is shown by inserting $p = 2+\frac{1}{N}$ into (2.29).

■

## 2.2 Proof of Theorem 1.5

Theorem 1.5 is proven in four steps:

### 2.2.1 Step 1: proof of the representation (1.25)

Using the Jordan decomposition $h(\kappa) = \sum_\mu \lambda_\mu(\kappa)\Pi_\mu(\kappa) + N_\mu(\kappa)$ and $\lambda_\mu(\kappa) = J(\kappa)+J_\mu(\kappa)$ (see (1.24)) gives:

$$e^{itJ(\kappa)}e^{-ith(\kappa)} = \sum_{\mu=1}^{S} e^{-itJ_\mu(\kappa)}\Pi_\mu(\kappa)\Big(\mathbb{1} + e^{-itN_\mu(\kappa)} - \mathbb{1}\Big) \tag{2.35}$$

for all $\kappa \in U_\varepsilon(0)$ and $t\in\mathbb{C}$. By (1.21) one has

$$e^{-itJ_\mu(\kappa)} = \sum_{k=0}^{\widetilde{N}-1} \frac{1}{k!}\,(-itJ_\mu(\kappa))^k + \exp_{\widetilde{N}}(-itJ_\mu(\kappa))\,. \tag{2.36}$$

Combining (2.35) and (2.36) gives

$$e^{itJ(\kappa)}e^{-ith(\kappa)} \;=\; E_{\widetilde{N}}(\kappa,t) + R_{\widetilde{N}}(\kappa,t) \qquad (\kappa\in U_\varepsilon(0),\ t\in\mathbb{C})\,, \tag{2.37}$$

where

$$E_{\widetilde{N}}(\kappa,t) := \sum_{\mu=1}^{S}\Big(\sum_{k=0}^{\widetilde{N}-1}\frac{1}{k!}\,(-it)^k J_\mu(\kappa)^k\Big)\Pi_\mu(\kappa)(\mathbb{1} + e^{-itN_\mu(\kappa)} - \mathbb{1}\,) \tag{2.38}$$

and $R_{\widetilde{N}}(\kappa,t)$ is given by (1.27).

Note that

$$e^{-itN_\mu(\kappa)} - \mathbb{1} \;=\; \sum_{n=1}^{m(\lambda_\mu)-1}\frac{1}{n!}\,(-it)^n N_\mu(\kappa)^n = \sum_{n=1}^{N-1}\frac{1}{n!}\,(-it)^n N_\mu(\kappa)^n\,, \tag{2.39}$$

where we have used that the $N_\mu(\kappa)$ are nilpotent (i.e., $N_\mu(\kappa)^n = 0$ for $n \ge m(\lambda_\mu)$) and the fact that $m(\lambda_\mu) \le N$.

Inserting (2.39) into (2.38) represents $E_{\widetilde N}$ as a product of two polynomials in $t$ of degree $\widetilde N - 1$ and $N-1$. Thus using $\sum_{\mu=1}^{S} \Pi_\mu(\kappa) = \mathbb{1}$ gives

$$E_{\widetilde N}(\kappa, t) =: \mathbb{1} + F_{\widetilde N}(\kappa, t) \qquad (\kappa \in U_\varepsilon(0),\ t \in \mathbb{C})\,, \tag{2.40}$$

where

$$F_{\widetilde N}(\kappa, t) = \sum_{k=1}^{\widetilde N + N - 2} \frac{1}{k!}\,(-it)^k F_k(\kappa) \tag{2.41}$$

for some $F_k(\kappa)$. So (2.38), (2.40) and (2.41) give (1.26). Then combining (2.37) and (2.40) proves (1.25).

■

### 2.2.2 Step 2: proof of the analyticity of $R_{\widetilde N}$ and $F_{\widetilde N}$ in both variables

Analyticity of $R_{\widetilde N}$ in the variable $t$: By (2.41), $F_{\widetilde N}(\kappa, \cdot)$ is analytic as a polynomial in $t$. Then from (1.25) it follows that $R_{\widetilde N}(\kappa, \cdot)$ is a sum of entire functions. Thus $R_{\widetilde N}(\kappa, \cdot)$ is an entire function.
We shall now prove the analyticity of $R_{\widetilde N}$ and $F_{\widetilde N}$ in the variable $\kappa$. For this we need the following

**Lemma 2.5** Assume (H1), (H2), (WLG2) and the notation of Theorem 1.5. Let $p$ be the number found in Remark 2.3. Fix $t \in \mathbb{C}$. Then, for all $\kappa \in U_\varepsilon(0)$ and $\mu \in \{1, \ldots, S\}$,

$$\Pi_\mu(\kappa)\, e^{-itN_\mu(\kappa)} \;=\; O(|\kappa|^{(-p+2)(N-1)}) \qquad (\kappa \to 0)\,, \tag{2.42}$$

$$\exp_{\widetilde N}(-itJ_\mu(\kappa)) \;=\; O(|\kappa|^{(2+\frac{1}{N})\widetilde N}) \qquad (\kappa \to 0)\,. \tag{2.43}$$

**Proof of Lemma 2.5:** For fixed $t$, combining (2.39), (2.33) and Proposition 2.2 proves (2.42). Estimate (2.43) is shown as follows:

$$\begin{aligned} \exp_{\widetilde N}(-itJ_\mu(\kappa)) &\overset{(1.21)}{=} J_\mu(\kappa)^{\widetilde N}(-it)^{\widetilde N}\,\widetilde{\exp}_{\widetilde N}(-itJ_\mu(\kappa)) \\ &\overset{\substack{(1.22)\\(2.27)}}{=} O(|\kappa|^{(2+\frac{1}{N})\widetilde N}) \qquad (\kappa \to 0,\ t \text{ fixed})\,. \end{aligned} \tag{2.44}$$

This finishes the proof of the lemma.

□

$R_{\widetilde N}(\kappa, t)$ is defined in (1.27). By Theorem B.8, all $\lambda_\mu(\cdot)$ (and thus $J_\mu(\cdot)$), $\Pi_\mu(\cdot)$, $N_\mu(\cdot)$ are (branches of one or more multi-valued) analytic functions (in $U_\varepsilon(0)$). Under analytic continuation around zero all $J_\mu(\cdot)$ ($\mu \in \{1, \ldots, S\}$) transform one into another, and so do the $\Pi_\mu(\cdot)$ and $N_\mu(\cdot)$. Since in the definition (1.27) of $R_{\widetilde N}(\cdot, t)$ we sum over all possible branches, $R_{\widetilde N}(\cdot, t)$ is single valued under analytic continuation around zero. Thus $R_{\widetilde N}(\cdot, t)$ is meromorphic in $U_\varepsilon(0)$ (and analytic in $U_\varepsilon(0)\backslash\{0\}$). Lemma 2.5 (and (1.27)) shows that, for $\widetilde N$ sufficiently large, $R_{\widetilde N}(\cdot, t)$ is bounded in all of $U_\varepsilon(0)$. Then, by Riemann's Removable Singularity Theorem, $R_{\widetilde N}(\cdot, t)$ is analytic in $U_\varepsilon(0)$. By (1.27), $F_{\widetilde N}(\cdot, t)$ is analytic in $U_\varepsilon(0)$, for $\widetilde N$ sufficiently large. Thus the functions $F_k(\cdot)$ in (2.41) are analytic. This proves the analyticity of $F_{\widetilde N}(\cdot, t)$ and $R_{\widetilde N}(\cdot, t)$ for all $\widetilde N$.

■

### 2.2.3 Step 3: estimates on $R_{\widetilde{N}}$

For $\kappa \in U_\varepsilon(0)$ and $t \in \mathbb{C}$ one gets, as $\kappa \to 0$,

$$
\begin{aligned}
R_{\widetilde{N}}(\kappa,t) &\overset{(1.27)}{\underset{(1.21)}{=}} \sum_{\mu=1}^{S}(-it)^{\widetilde{N}} J_\mu(\kappa)^{\widetilde{N}} \widetilde{\exp}_{\widetilde{N}}(-itJ_\mu(\kappa))\,(\Pi_\mu(\kappa) + e^{-itN_\mu(\kappa)} - \mathbb{1}\,) \\
&\overset{(2.39)}{=} \sum_{\mu=1}^{S} \widetilde{\exp}_{\widetilde{N}}(-itJ_\mu(\kappa))(-it)^{\widetilde{N}} J_\mu(\kappa)^{\widetilde{N}} \Big(\Pi_\mu(\kappa) + \\
&\qquad \sum_{j=1}^{N-1} \frac{1}{j!}\,(-it)^j N_\mu(\kappa)^j \Pi_\mu(\kappa)\Big) \\
&\overset{(2.28)}{=} \sum_{\mu=1}^{S} \widetilde{\exp}_{\widetilde{N}}(-itJ_\mu(\kappa))\Big(\sum_{j=0}^{N-1} \frac{1}{j!}\,(-it)^{\widetilde{N}+j} O(|\kappa|^{2j}) J_\mu(\kappa)^{\widetilde{N}} \Pi_\mu(\kappa)\Big).
\end{aligned}
\tag{2.45}
$$

Choosing

$$\widetilde{N} = (p-2)(N-1)N \tag{2.46}$$

gives

$$J_\mu(\kappa)^{\widetilde{N}} \Pi_\mu(\kappa) = O(|\kappa|^{2\widetilde{N}}) \qquad (\kappa \to 0)\,, \tag{2.47}$$

which is a consequence of (2.29). Then

$$
\begin{aligned}
\|R_{\widetilde{N}}(\kappa,t)\| &\overset{(2.47)}{\underset{(1.22)}{\leq}} \sum_{\mu=1}^{S} \frac{1}{\widetilde{N}!}\, e^{|t|\,|J_\mu(\kappa)|} \Big(\sum_{j=0}^{N-1} \frac{1}{j!}\, |t|^{\widetilde{N}} |t|^j |\kappa|^{2j} O(|\kappa|^{2\widetilde{N}})\Big) \\
&\leq \quad e^{c\,|t|\,|\kappa|^{2+\frac{1}{N}}} \sum_{j=0}^{N-1} |t|^{j+\widetilde{N}} O(|\kappa|^{2(j+\widetilde{N})})\,,
\end{aligned}
$$

where $c$ is some strictly positive constant. This is (1.29).

■

### 2.2.4 Step 4: estimates on $F_{\widetilde{N}}$

Analyticity of $F_{\widetilde{N}}(\kappa,\cdot)$ and (2.41) imply

$$(-i)^k F_k(\kappa) = \frac{d^k}{dt^k}\Big|_{t=0} F_{\widetilde{N}}(\kappa,t) \qquad (\kappa \in U_\varepsilon(0),\ k \in \mathbb{N})\,. \tag{2.48}$$

(We remark that, by (2.41), $F_k(\cdot) = 0$ for $k > \widetilde{N} + N - 2$ and for $k = 0$.) Next we will prove

$$\frac{d^k}{dt^k}\Big|_{t=0} F_{\widetilde{N}}(\kappa,t) = O(|\kappa|^{2k}) \qquad (\kappa \to 0\,,\ k \in \{1,\ldots,\widetilde{N}+N-2\})\,, \tag{2.49}$$

where $\widetilde{N} \geq (p-2)(N-1)N$. By (1.25), we get

$$F_{\widetilde{N}}(\kappa,t) = e^{itJ(\kappa)} e^{-ith(\kappa)} - R_{\widetilde{N}}(\kappa,t) - \mathbb{1}\,, \tag{2.50}$$

and we calculate, for $\kappa \in U_\varepsilon(0)$,

$$
\begin{aligned}
\frac{d^k}{dt^k}\Big|_{t=0} e^{itJ(\kappa)} e^{-ith(\kappa)} &= \frac{d^k}{dt^k}\Big|_{t=0} \exp\big(it(J(\kappa)\mathbb{1} - h(\kappa))\big) \\
&= \big(iJ(\kappa)\mathbb{1} - ih(\kappa)\big)^k \\
&\overset{(1.20)}{=} O(|\kappa|^{2k}) \qquad (\kappa \to 0\,,\ k \in \mathbb{N})
\end{aligned}
\tag{2.51}
$$

and

$$\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) = \sum_\mu \frac{d^k}{dt^k}\Big|_{t=0} \exp_{\widetilde{N}}(-itJ_\mu(\kappa))\,\Pi_\mu(\kappa)e^{-itN_\mu(\kappa)}$$
$$= \sum_\mu \sum_{n=0}^{k} \begin{pmatrix} k \\ n \end{pmatrix} \left(\frac{d^n}{dt^n}\Big|_{t=0} \exp_{\widetilde{N}}(-itJ_\mu(\kappa))\right)$$
$$\cdot \left(\frac{d^{k-n}}{dt^{k-n}}\Big|_{t=0} \Pi_\mu(\kappa)e^{-itN_\mu(\kappa)}\right), \tag{2.52}$$

where

$$\frac{d^k}{dt^k} \exp_{\widetilde{N}}(-itJ_\mu(\kappa)) \quad = \quad (-i)^k J_\mu(\kappa)^k \sum_{n=\widetilde{N}-k}^{\infty} \frac{1}{n!}\,(-itJ_\mu(\kappa))^n \tag{2.53}$$

and thus

$$\frac{d^k}{dt^k}\Big|_{t=0} \exp_{\widetilde{N}}(-itJ_\mu(\kappa)) = \begin{cases} 0 & , \quad k \in \{0,1,\dots,\widetilde{N}-1\} \\ (-i)^k J_\mu(\kappa)^k & , \quad k \geq \widetilde{N} \end{cases}. \tag{2.54}$$

Inserting (2.54) into (2.52) gives

$$\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) = 0 \qquad (k \in \{0,1,\dots,\widetilde{N}-1\}) \tag{2.55}$$

and, for $k \geq \widetilde{N}$,

$$\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) \quad = \quad \sum_\mu \sum_{n=\widetilde{N}}^{k} \begin{pmatrix} k \\ n \end{pmatrix} \left(\frac{d^n}{dt^n}\Big|_{t=0} \exp_{\widetilde{N}}(-itJ_\mu(\kappa))\right)$$
$$\left(\frac{d^{k-n}}{dt^{k-n}}\Big|_{t=0} \Pi_\mu(\kappa)e^{-itN_\mu(\kappa)}\right). \tag{2.56}$$

Using

$$\frac{d^k}{dt^k}\Big|_{t=0} e^{-itN_\mu(\kappa)} = (-i)^k N_\mu(\kappa)^k \qquad (k \in \mathbb{N})$$

yields

$$\frac{d^k}{dt^k}\Big|_{t=0} \Pi_\mu(\kappa)e^{-itN_\mu(\kappa)} = \begin{cases} \Pi_\mu(\kappa) & , \quad k = 0 \\ (-i)^k N_\mu(\kappa)^k & , \quad k \geq 1 \end{cases}, \tag{2.57}$$

where $N_\mu(\cdot)^k = 0$ $(k \geq N)$.

Then using (2.54) and (2.57) in (2.56) gives, for $k > \widetilde{N}$,

$$\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) \quad = \quad \sum_\mu \sum_{n=\widetilde{N}}^{k-1} \begin{pmatrix} k \\ n \end{pmatrix} (-i)^k J_\mu(\kappa)^n N_\mu(\kappa)^{k-n} + (-i)^k J_\mu(\kappa)^k \Pi_\mu(\kappa), \tag{2.58}$$

and

$$\frac{d^{\widetilde{N}}}{dt^{\widetilde{N}}}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) = \sum_\mu (-i)^{\widetilde{N}} J_\mu(\kappa)^{\widetilde{N}} \Pi_\mu(\kappa) \overset{(2.29)}{=} O(|\kappa|^{(-p+2)(N-1)+(2+\frac{1}{N})\widetilde{N}}) \tag{2.59}$$

as $\kappa \to 0$. In (2.59) choosing $\widetilde{N}$ as in (2.46) gives

$$\frac{d^{\widetilde{N}}}{dt^{\widetilde{N}}}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) = O(|\kappa|^{2\widetilde{N}}) \qquad (\kappa \to 0)\,. \tag{2.60}$$

From (2.58) it follows that, for $k > \widetilde{N}$,

$$\begin{aligned}
\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) &\overset{(2.28)}{=} (-i)^k \sum_{\mu} \Big( \sum_{n=\widetilde{N}}^{k-1} \binom{k}{n} O(|\kappa|)^{2(k-n)} J_\mu(\kappa)^n \Pi_\mu(\kappa) + \\
&\qquad\qquad J_\mu(\kappa)^k \Pi_\mu(\kappa) \Big) \\
&= (-i)^k \sum_{\mu} \sum_{n=\widetilde{N}}^{k} \binom{k}{n} O(|\kappa|)^{2(k-n)} J_\mu(\kappa)^n \Pi_\mu(\kappa) \\
&\overset{(2.29)}{=} \sum_{n=\widetilde{N}}^{k} \binom{k}{n} O(|\kappa|^{2(k-n)+(-p+2)(N-1)+(2+\frac{1}{N})n}) \\
&= O(|\kappa|^{2k+(-p+2)(N-1)+\frac{\widetilde{N}}{N}}) \qquad (\kappa \to 0)\,. \qquad (2.61)
\end{aligned}$$

Then choosing $\widetilde{N}$ as in (2.46), estimate (2.61) becomes

$$\frac{d^k}{dt^k}\Big|_{t=0} R_{\widetilde{N}}(\kappa,t) = O(|\kappa|^{2k}) \qquad (\kappa \to 0,\ k > \widetilde{N})\,. \tag{2.62}$$

Choosing $\widetilde{N}$ as in (2.46) and combining (2.48), (2.50), (2.51), (2.55), (2.60) and (2.62) proves (2.49).

Finally, using (2.49) in (2.41) proves (1.28). This finishes the proof of Theorem 1.5.

■

# 3 Special results: the extreme cases of eigenvalue splitting

In this section we use results and language of analytic perturbation theory, as explained in Appendix A and Appendix B. We recommend that the reader not familiar with this subject looks at these appendices first.

Assuming (H1), there are three extreme cases of splitting:

**(E1)** Under perturbation, the eigenvalue $\lambda_0$ of $h(0)$ splits into one $N$-cycle:

$$\sigma(h(\kappa)) = \{\lambda_{1,1}(\kappa), \lambda_{1,2}(\kappa), \ldots, \lambda_{1,N}(\kappa)\} \qquad (\kappa \in D_\varepsilon(0))\,.$$

**(E2)** Under perturbation, the eigenvalue $\lambda_0$ of $h(0)$ splits into $N$ 1-cycles:

$$\sigma(h(\kappa)) = \big\{\{\lambda_{1,1}(\kappa)\}, \ldots, \{\lambda_{N,1}(\kappa)\}\big\} \qquad (\kappa \in U_\varepsilon(0))\,.$$

**(E3)** Under perturbation, the eigenvalue $\lambda_0$ of $h(0)$ "splits" into one 1-cycle (i.e., no splitting and permanent degeneracy[7]):

$$\sigma(h(\kappa)) = \{\lambda_{1,1}(\kappa)\} \qquad (\kappa \in U_\varepsilon(0))\,.$$

[7]See [K2, II §1.1] for the definition of *permanently degenerate*.

## 3.1 Analyzing (E1)

**Lemma 3.1** Assume (H1) and (E1). Then the factorization (B.14) is

$$\chi(\kappa,\lambda)=\chi_1(\kappa,\lambda)\,,\qquad n_1=N\,,\qquad m_1=1\,, \tag{3.1}$$

and the spectrum $\sigma(h(\cdot))$, the (pairwise distinct) eigenvalues $\lambda_{1,j}(\cdot)$, the eigenprojections $\Pi_{1,j}(\cdot)$ and the eigennilpotents $N_{1,j}(\cdot)$ satisfy

$$\sigma(h(\kappa)) \;=\; \{\lambda_{1,1}(\kappa),\lambda_{1,2}(\kappa),\ldots,\lambda_{1,N}(\kappa)\}\,, \tag{3.2}$$

$$\lambda_{1,j}(\kappa) \;=\; \sum_{\nu=0}^{\infty} a_\nu^{(1)} e^{\frac{2\pi i}{N}\cdot\nu\cdot(j-1)}\kappa^{\nu/N} \quad \text{of multiplicity} \quad m(\lambda_{1,j}(\kappa))=1\,,\quad a_0^{(1)}=\lambda_0\,, \tag{3.3}$$

$$\Pi_{1,j}(\kappa) \;=\; \sum_{\nu=\nu_0}^{\infty} \Pi_\nu^{(1,j)}\,\kappa^{\nu/N} \qquad \text{for some } \nu_0<0\,, \tag{3.4}$$

$$N_{1,j}(\kappa) \;=\; 0 \tag{3.5}$$

for all $\kappa\in D_\varepsilon(0)$, $j\in\{1,\ldots,N\}$, some $a_\nu^{(1)}\in\mathbb{C}$ and some $\Pi_\nu^{(1,j)}\in\mathrm{End}(\mathscr{H}_0)$, $\Pi_{\nu_0}^{(1,j)}\neq 0$.

**Proof**: (3.1) is clear from Remark B.4. Equations (3.2) and (3.3) follow from Proposition B.3. Then (3.4) is a consequence of Theorem B.8 (the branching order of $\lambda_{1,j}(\cdot)$ and $\Pi_{1,j}(\cdot)$ coincides), and (3.5) is due to the fact that all eigenvalues $\lambda_{1,j}(\kappa)$ for $\kappa\neq 0$ are of multiplicity one. That $\Pi_{\nu_0}^{(1,j)}\neq 0$ for some $\nu_0<0$ follows from Butler's Theorem (Theorem B.9). ■

**Lemma 3.2** Assume (H1), (H2) and (E1). Let $k_1\in\mathbb{N}$ be the branching generation of $\lambda_{1,j}(\cdot)$ introduced in Definition B.10. Then $k_1\geq 2$ and we may rewrite (3.3) as:

$$\lambda_{1,j}(\kappa) \;=\; J_1^{(k_1)}(\kappa)+J_{1,j}^{(k_1,\infty)}(\kappa)\,, \tag{3.6}$$

$$J_1^{(k_1)}(\kappa) \;=\; \lambda_0+\sum_{\nu=1}^{k_1} a_{N\cdot\nu}^{(1)}\kappa^\nu\,, \tag{3.7}$$

$$J_{1,j}^{(k_1,\infty)}(\kappa) \;=\; \sum_{\nu=1}^{\infty} a_{Nk_1+\nu}^{(1)} e^{\frac{2\pi i}{N}\cdot(Nk_1+\nu)\cdot(j-1)}\,\kappa^{k_1+\frac{\nu}{N}} \tag{3.8}$$

for all $j\in\{1,\ldots,N\}$ and $\kappa\in D_\varepsilon(0)$.

In particular,

**Corollary 3.3** (H1), (H2) and (E1) imply (WLG2), where $\alpha=a_N^{(1)}$ and $\beta=a_{2N}^{(1)}$ are introduced in (3.3).

**Proof of Lemma 3.2**: Since $\lambda_0$ splits int one $N$-cycle, all eigenvalues $\lambda_{1,j}(\cdot)$ transform one into another under analytic continuation; cf. Remark B.4. Thus they all have the same branching generation $k_1$ (cf. Definition B.10) and the same $(k_1)$-jet $J_1^{(k_1)}(\,\cdot\,)$. By (H1) and (H2) it follows from (twice applying) the reduction process described in Section 1 that $k_1\geq 2$. ■

**Theorem 3.4** Assume (H1), (H2) and (E1). Assume the notation of Lemma 3.2 and Lemma 3.1. Then, for $t \in \mathbb{C}$ and $\kappa \in U_\varepsilon(0)$,

$$
\begin{aligned}
e^{-ith(\kappa)} &= e^{-itJ_1^{(k_1)}(\kappa)}(\mathbb{1} + F_{\widetilde{N}}(\kappa,t) + R_{\widetilde{N}}(\kappa,t))\,, && (3.9)\\
F_{\widetilde{N}}(\kappa,t) &:= \sum_{\nu=1}^{\widetilde{N}-1} \frac{(-it)^\nu}{\nu!} F_\nu(\kappa)\,, \qquad F_\nu(\kappa) := \sum_{j=1}^{N} \big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^\nu \Pi_{1,j}(\kappa)\,, && (3.10)\\
R_{\widetilde{N}}(\kappa,t) &:= \sum_{j=1}^{N} \exp_{\widetilde{N}}(-itJ_{1,j}^{(k_1,\infty)}(\kappa))\,\Pi_{1,j}(\kappa). && (3.11)
\end{aligned}
$$

(The function $\exp_{\widetilde{N}}$ is defined in (1.21).) $F_{\widetilde{N}}$ and $R_{\widetilde{N}}$ are analytic in both variables, $\kappa \in U_\varepsilon(0)$ and $t \in \mathbb{C}$. For $\widetilde{N} \geq (k_1 + \frac{1}{N} - 2)(N-1)N$ and $\kappa \in U_\varepsilon(0)$ the remainders satisfy

$$
\begin{aligned}
&F_\nu(\kappa) = O(|\kappa|^{2\nu}) \qquad (\nu \in \{1, \ldots, \widetilde{N}-1\})\,, && (3.12)\\
&\|R_{\widetilde{N}}(\kappa,t)\| \leq O(|t\kappa^{k_1}|^{\widetilde{N}}) \exp\big(c'|t| \cdot |\kappa|^{k_1+\frac{\gamma}{N}}(1 + O(|\kappa|^{\frac{1}{N}}))\big) && (3.13)
\end{aligned}
$$

as $\kappa \to 0$, uniformly for $t \in \mathbb{C}$. In (3.13), $\gamma$ is some natural number in $[1, N-1]$, such that $Nk_1 + \gamma$ is the lowest order in which a nonzero coefficient $a_{Nk_1+\nu}^{(1)}$ (introduced in (3.8)) appears, and

$$
c' := \max_{j \in \{1,\ldots,N\}} |c_j|\,, \qquad c_j := a_{Nk_1+\gamma}^{(1)} e^{2\pi i(k_1 + \frac{\gamma}{N})(j-1)}\,. \tag{3.14}
$$

**Proof**: See Section 3.1.1.

**Theorem 3.5** Assume (H1), (H2), (H3) and (E1). Let $\gamma$ be as in Theorem 3.4. Then:

(i) There exists a unique $k_* \in \mathbb{N}$ with $2 \leq k_* \leq k_1$, such that for all $\kappa \in D_\varepsilon(0) \cap \mathbb{R}$ and $j \in \{1, \ldots, N\}$

$$
|\mathrm{Im}\lambda_{1,j}(\kappa)| = c\,|\kappa|^{k_*}(1 + O(|\kappa|^{1/N})) \qquad (\kappa \to 0)\,, \tag{3.15}
$$

where

$$
c := |\mathrm{Im}\, a_{Nk_*}^{(1)}| > 0 \tag{3.16}
$$

with $a_{Nk_*}^{(1)}$ introduced in (3.7), $\lambda_{1,j}(\cdot)$ is given by (3.6) and, for $\kappa < 0$, $\lambda_{1,j}(\kappa)$ denotes any analytic continuation of $\lambda_{1,j}(\cdot)$ (originally defined in $D_\varepsilon(0)$).[8]

(ii) For $J_1^{(k_1)}(\kappa)$ defined as in (3.7) one has, for $\kappa \in U_\varepsilon(0) \cap \mathbb{R}$ and $t \geq 0$,

$$
|e^{-itJ_1^{(k_1)}(\kappa)}| \leq e^{-ct|\kappa|^{k_*}(1+O(|\kappa|))} \qquad (\kappa \to 0) \tag{3.17}
$$

with $c$ given by (3.16). In particular, the function $e^{-ith(\kappa)}$ (given by (3.9)) satisfies: For $\kappa \in U_\varepsilon(0) \cap \mathbb{R}$ and $\widetilde{N} \geq (k_1 + \frac{1}{N} - 2)(N-1)N$

$$
\begin{aligned}
\|e^{-ith(\kappa)}\| \leq e^{-ct|\kappa|^{k_*}(1+O(|\kappa|))}\Big(1 &+ \sum_{\nu=1}^{\widetilde{N}-1} O(|\kappa^2 t|^\nu)\\
&+ O(|t\kappa^{k_1}|^{\widetilde{N}})\, e^{c'|t|\,|\kappa|^{k_1+\frac{\gamma}{N}}(1+O(|\kappa|^{\frac{1}{N}}))}\Big)
\end{aligned} \tag{3.18}
$$

as $\kappa \to 0$, uniformly for $t \in [0, \infty)$. The constants $c'$ and $c$ are given by (3.14) and (3.16).

**Proof**: See Section 3.1.1.

[8] We choose $k_*$ as the minimal index $\nu$ such that $\mathrm{Im}\, a_{N\nu}^{(1)} \neq 0$. This gives (3.15). The point is to show that the set of such $\nu$'s is not empty.

### 3.1.1 Proof of Theorem 3.4 and Theorem 3.5

By Corollary 3.3, (WLG2) holds. We shall now sharpen some of the estimates in the proof of Theorem 1.5, using (E1). By Corollary 1.4, one has

**Corollary 3.6** Assume (H1), (H2) and (E1). Then, for $\kappa \in U_\varepsilon(0)$,

$$h(\kappa) - J_1^{(k_1)}(\kappa)\mathbb{1}_{\mathscr{H}_0} = O(|\kappa|^2) \qquad (\kappa \to 0)\,, \tag{3.19}$$

where $k_1 \geq 2$ is the branching generation (cf. Lemma 3.2) and $J_1^{(k_1)}(\kappa)$ is given by (3.7).

**Remark 3.7** Assume (H1) and (H2). If the eigenvalues $\lambda_{1,j}(\cdot)$ ($j \in \{1, \dots, N\}$) of $h(\cdot)$ build one $N$-cycle splitting from $\lambda_0$ with branching generation $k_1$, then there exist some diagonalizable $h^{(\nu_0)}(0)$, where $\nu_0 \in \{1, 2, \dots, k_1 - 1\}$. If all $h^{(\nu)}(0)$ ($\nu \in \{0, 1, \dots, k_1 - 1\}$) are diagonalizable (the extreme case), one has a better estimate than (3.19):

$$h(\kappa) - J_1^{(k_1)}(\kappa)\mathbb{1}_{\mathscr{H}_0} = O(|\kappa|^{k_1}) \qquad (\kappa \to 0)\,. \tag{3.20}$$

However, in general one does not know more than (3.19), as the following instructive example shows: For $\kappa \in U_\varepsilon(0)$ let

$$h(\kappa) = \begin{pmatrix} 0 & \kappa^2 \\ \kappa^q & 0 \end{pmatrix} = \kappa^2 \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \kappa^{q-2} & 0 \end{pmatrix} \right), \tag{3.21}$$

where $q$ is odd with $q \geq 3$ (since $h'(0)$ is diagonalizable). Then

$$\lambda_{1,j}(\kappa) = (-1)^{j-1}\sqrt{\kappa^{q+2}} \qquad (j \in \{1, 2\},\ \kappa \in D_\varepsilon(0))\,, \tag{3.22}$$

$k_1 = 2$ and $J_1^{(k_1)}(\kappa) = 0$. Thus, taking $q$ large, $\lambda_{1,j}(\kappa)$ vanishes to arbitrarily large order as $\kappa \to 0$, while $h(\kappa) - J_1^{(k_1)}(\kappa)\mathbb{1}_{\mathscr{H}_0}$ does not satisfy any better estimate than (3.19).

**Proposition 3.8** Assume (H1), (H2) and (E1). Then the eigenprojections $\Pi_{1,j}(\cdot)$ have convergent Puiseux-Laurent expansions (around zero). More precisely, there is

$$\mathbb{Z} \ni \nu_0 \geq N\Big( - (N-1)k_1 + 2N + \frac{1}{N} - 3\Big)\,, \tag{3.23}$$

such that

$$\Pi_{1,j}(\kappa) = \sum_{\nu=\nu_0}^{\infty} \Pi_\nu^{(1,j)} \kappa^{\frac{\nu}{N}} \tag{3.24}$$

for some $\Pi_\nu^{(1,j)} \in \mathrm{End}(\mathscr{H}_0)$, $\Pi_{\nu_0}^{(1,j)} \neq 0$ and all $\kappa \in D_\varepsilon(0)$. Here $k_1 \geq 2$ denotes the branching generation of $\lambda_{1,j}(\cdot)$ according to Definition B.10. In particular,

$$\Pi_{1,j}(\kappa) = O(|\kappa|^{-(N-1)k_1 + 2N + \frac{1}{N} - 3}) \qquad (\kappa \to 0) \tag{3.25}$$

for $j \in \{1, \dots, N\}$, $\kappa \in D_\varepsilon(0)$.

**Proof**: By Corollary 3.3, Proposition 2.2 is applicable, and (2.15) gives $p = k_1 + \frac{1}{N}$. Then substituting $p = k_1 + \frac{1}{N}$ into (2.12) proves (3.25). Butler's Theorem (Theorem B.9) and the estimate (3.25) give (3.23) and (3.24).

■

**Corollary 3.9** Assume (H1), (H2) and (E1). Then for $j \in \{1, \dots, N\}$, $\kappa \in D_\varepsilon(0)$ and $n \in \mathbb{N}\backslash\{0\}$

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n = O(|\kappa|^{(k_1+\frac{1}{N})n}) \qquad (\kappa \to 0)\,, \tag{3.26}$$

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n \Pi_{1,j}(\kappa) = O(|\kappa|^{k_1(n-N+1)+\frac{n+1}{N}+2N-3}) \qquad (\kappa \to 0)\,, \tag{3.27}$$

where $J_{1,j}^{(k_1,\infty)}(\kappa)$ and $\Pi_{1,j}(\kappa)$ are given by (3.8) and (3.4). In particular:

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n \Pi_{1,j}(\kappa) = O(|\kappa|^2) \qquad (n \geq N-1)\,, \tag{3.28}$$

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n \Pi_{1,j}(\kappa) = O(|\kappa|^{k_1 n}) \qquad \big(n \geq (k_1 + \frac{1}{N} - 2)(N-1)N\big)\,, \tag{3.29}$$

as $\kappa \to 0$, independent of the size of $k_1$.

**Proof**: Combining (3.8) and (3.25) proves (3.27). For $n \geq N-1$ and $N \geq 2$ the exponent in (3.27) obeys the following estimate:

$$k_1(n-N+1) + \frac{n+1}{N} + 2N - 3 \geq k_1(N-1-N+1) + 2N - 2 = 2N-2 \geq 2\,.$$

This proves (3.28). In the estimate (3.27) setting $n = (k_1 + \frac{1}{N} - 2)(N-1)N$ gives the exponent

$$k_1(n-N+1) + \frac{n+1}{N} + 2N - 3 = k_1 n\,.$$

This proves (3.29).

■

**Remark 3.10** In general (depending on the size of $k_1$) for $n < N-1$ the expression $\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n \Pi_{1,j}(\kappa)$ has a singularity at $\kappa = 0$. However, if $k_1 = 2$, then for all $\dim \mathscr{H}_0 = N \in \mathbb{N}$ and all $n \in \{1, \dots, N-2\}$ one has (for $n \geq N-1$ one has the better estimate (3.28))

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^n \Pi_{1,j}(\kappa) \quad = \quad O(1) \qquad (\kappa \to 0)\,,$$

which can be seen as follows: The inequality $k_1(n-N+1) + \frac{n+1}{N} + 2N - 3 \geq 0$ excludes negative powers of $|\kappa|$ on the r.h.s. of (3.27). Setting $k_1 = 2$, this inequality is equivalent to $2(n+1) + \frac{n+1}{N} - 3 \geq 0$, which obviously holds for $n \geq 1$.

**Proof of Theorem 3.4**: Due to Corollary 3.3, in this special case Theorem 1.5 applies. Following the proof of Theorem 1.5 (using Lemma 3.2) one checks that $J(\cdot)$ may be replaced by $J^{(k_1)}(\cdot)$ and $J_\mu(\cdot)$ by $J_{1,j}^{(k_1,\infty)}(\cdot)$ (where $J^{(k_1)}$ and $J_{1,j}^{(k_1,\infty)}$ are defined in (3.7) and (3.8)). Furthermore, in Theorem 1.5 we may set $S = N$, $\Pi_\mu(\cdot) = \Pi_{1,j}(\cdot)$ and $N_\mu(\cdot) = N_{1,j}(\cdot) \overset{(3.5)}{=} 0$. Then (1.26) becomes

$$F_{\widetilde{N}}(\kappa, t) = \sum_{\nu=1}^{\widetilde{N}-1} \frac{(-it)^\nu}{\nu!} \sum_{j=1}^{N} \big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^\nu \Pi_{1,j}(\kappa), \tag{3.30}$$

where we have used that $\sum_{j=1}^{N}\Pi_{1,j}(\kappa)=\mathbb{1}$ . This proves (3.9), (3.10) and (3.11). Furthermore, in Theorem 1.5 one can set $p=k_1+\frac{1}{N}$, which is a consequence of Remark 2.3 and Lemma 3.2. As in the proof of Theorem 1.5, we choose

$$\widetilde{N}\geq(p-2)(N-1)N=(k_1+\frac{1}{N}-2)(N-1)N. \tag{3.31}$$

Then equations (2.48) and (2.49) (which appear in the proof of Theorem 1.5) give

$$(-i)^\nu F_\nu(\kappa)=O(|\kappa|^{2\nu})\qquad(\kappa\to0\,,\ \nu\in\{1,\dots,\widetilde{N}+N-2\}).$$

This proves (3.12). Using (1.21) in (3.11) gives

$$R_{\widetilde{N}}(\kappa,t)=(-it)^{\widetilde{N}}\sum_{j=1}^{N}\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^{\widetilde{N}}\Pi_{1,j}(\kappa)\,\widetilde{\exp}_{\widetilde{N}}(-itJ_{1,j}^{(k_1,\infty)}(\kappa)).$$

Then, by (1.22),

$$\|R_{\widetilde{N}}(\kappa,t)\|\leq|t|^{\widetilde{N}}\sum_{j=1}^{N}\frac{1}{\widetilde{N}!}\,e^{|tJ_{1,j}^{(k_1,\infty)}(\kappa)|}\,\|\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^{\widetilde{N}}\Pi_{1,j}(\kappa)\|\,. \tag{3.32}$$

By (3.8),

$$e^{|tJ_{1,j}^{(k_1,\infty)}(\kappa)|}=e^{|t|\,|c_j|\,|\kappa|^{k_1+\frac{\gamma}{N}}(1+O(|\kappa|^{\frac{1}{N}}))}\qquad(\kappa\to0)\,. \tag{3.33}$$

For $\widetilde{N}$ as in (3.31), (3.29) implies

$$\big(J_{1,j}^{(k_1,\infty)}(\kappa)\big)^{\widetilde{N}}\Pi_{1,j}(\kappa)=O(|\kappa|^{k_1\widetilde{N}})\qquad(\kappa\to0)\,. \tag{3.34}$$

Finally, combining (3.34), (3.33) and (3.32) proves (3.13), which finishes the proof of the theorem.

■

**Proof of Theorem 3.5, ad (i)**: We show that there exists $\nu\leq k_1$ with $\mathrm{Im}a_{N\nu}^{(1)}\neq0$. Assume that this is not the case. Then $J_1^{(k_1)}(\kappa)$ in (3.7) is real for $\kappa\in\mathbb{R}$, and $\mathrm{Im}\,\lambda_{1,j}(\kappa)=\mathrm{Im}\,J_{1,j}^{(k_1,\infty)}(\kappa)$. By (3.8), we have for $j\in\{1,\dots,N\}$

$$J_{1,j}^{(k_1,\infty)}(\kappa)\quad=\quad c_j\kappa^{k_1+\frac{\gamma}{N}}(1+O(|\kappa|^{\frac{1}{N}}))\qquad(\kappa\to0)\,, \tag{3.35}$$

where $\gamma$ and $c_j$ are as in Theorem 3.4. It follows that there exist $j,j'\in\{1,\dots,N\}$ for which $\mathrm{Im}J_{1,j}^{(k_1,\infty)}(\kappa)$ and $\mathrm{Im}J_{1,j'}^{(k_1,\infty)}(\kappa)$ do not vanish and have opposite signs for some value of $\kappa\in\mathbb{R}$. For $N\geq3$, this occurs for any $\kappa>0$. For $N=2$, this may not occur for $\kappa>0$ (if $a_{Nk_1+1}^{(1)}$ is real), but then it necessarily occurs for $\kappa<0$ by analytic continuation. This contradicts (H3). Thus there exists $\nu\leq k_1$ with (3.15), and by Lemma 1.2 we have $\nu\geq2$. Then the minimal value of such $\nu$ gives the unique value of $k_*$ with (3.15).

□

**ad (ii)**: (3.7) and $k_1\geq k_*\in\mathbb{N}$ imply (3.17). Combining (3.17), (3.13), (3.12), (3.10) and (3.9) proves (3.18).

■

## 3.2 Analyzing (E2)

Assume (H1), (H2) and (E2). Then the factorization (B.14) is

$$\chi(\kappa,\lambda)=\prod_{\ell=1}^{N}\chi_\ell(\kappa,\lambda)\,,\qquad n_\ell=m_\ell=1 \tag{3.36}$$

and

$$\sigma(h(\kappa)) = \big\{\{\lambda_{1,1}(\kappa)\},\dots,\{\lambda_{N,1}(\kappa)\}\big\}\,, \tag{3.37}$$

$$\lambda_{\ell,1}(\kappa) = \sum_{\nu=0}^{\infty} a_\nu^{(\ell)}\kappa^\nu \quad \text{of multiplicity} \quad m(\lambda_{\ell,1}(\kappa)) = 1, \quad a_0^{(\ell)} = \lambda_0, \quad a_1^{(\ell)} \text{ real}, \tag{3.38}$$

$$\Pi_{\ell,1}(\kappa) = \sum_{\nu=\nu_0}^{\infty} \Pi_\nu^{(\ell,1)}\,\kappa^\nu \quad \text{for some } \nu_0 \in \mathbb{Z}\,, \tag{3.39}$$

$$N_{\ell,1}(\kappa) = 0 \tag{3.40}$$

for all $\ell \in \{1,\dots,N\}$, $\kappa \in U_\varepsilon(0)$, some $a_\nu^{(\ell)} \in \mathbb{C}$ and some $\Pi_\nu^{(\ell,1)} \in \mathrm{End}(\mathscr{H}_0)$. For $t \in \mathbb{C}$ and $\kappa \in U_\varepsilon(0)$, one has

$$e^{-ith(\kappa)} = \sum_{\ell=1}^{N} e^{-it\lambda_{\ell,1}(\kappa)}\Pi_{\ell,1}(\kappa)\,, \tag{3.41}$$

where the single projections $\Pi_{\ell,1}(\cdot)$ may have a pole at zero.

In this extreme case of eigenvalue splitting, in general, one does not get any better estimates than those contained in Theorem 1.5 and Corollary 1.6, which can be seen as follows:

First, for $\kappa \neq 0$, partition $\sigma(h(\kappa))$ into clusters $\mathcal{C}_k$ ($k \in \{1,\dots,R\}$ for some $R \leq N$), which are maximal in quadratic order. Fix $k$ and let $\lambda_{\ell,1}(\cdot) \in \mathcal{C}_k$. For $\ell \neq \ell'$ define $k_{\ell,\ell'} := \min\{\nu \,|\, a_\nu^{(\ell)} \neq a_\nu^{(\ell')}\}$ and $k_\ell^* := \max\{k_{\ell,\ell'} \,|\, 1 \leq \ell' \leq N\}$. Then there exists a contour $\Gamma_\ell(\kappa)$, separating $\lambda_{\ell,1}(\kappa)$ from $\sigma(h(\kappa))\backslash\{\lambda_{\ell,1}(\kappa)\}$, of the form

$$z = \lambda_{\ell,1}(\kappa) + \rho_\ell|\kappa|^{k_\ell^*}, \qquad \mathrm{dist}(\Gamma_\ell(\kappa),\sigma(h(\kappa))) \geq \rho_\ell|\kappa|^{k_\ell^*}\,. \tag{3.42}$$

Then following the proof of Proposition 2.2, with $p$ replaced by $k_\ell^*$, one finds

$$\Pi_{\ell,1}(\kappa) = O(|\kappa|^{\nu_\ell}) \qquad (\kappa \to 0)\,, \qquad\qquad \nu_\ell := (2-k_\ell^*)(\mathcal{N}_\ell - 1)\,, \tag{3.43}$$

where $\mathcal{N}_\ell = \#\mathcal{C}_k$ for the cluster $\mathcal{C}_k$ with $\lambda_{\ell,1}(\cdot) \in \mathcal{C}_k$. (# denotes cardinality.)

If $\sigma(h(\kappa))$ does not split into $N$ clusters of cardinality 1, which are maximal in quadratic order, in general one cannot do better than Theorem 1.5, since in general only the quadratic jet $J(\kappa)$ is common to all eigenvalues, after adiabatic reduction to (WLG2), but in clusters of three or more eigenvalues, $k_\ell^*$ may be arbitrarily large. The absence of nilpotent terms does not help in improving the estimates in Theorem 1.5 (and Corollary 1.6).

Of course, if there are only few clusters $\mathcal{C}_k$ maximal in quadratic order with $\#\mathcal{C}_k > 1$ and $k_\ell^* = 3, 4, \dots$, one can compute more precisely polynomial corrections. We leave this to the interested reader.

## 3.3 Analyzing (E3)

**Theorem 3.11** Assume (H1) and (E3). Then the factorization (B.14) is

$$\chi(\kappa,\lambda) \;=\; \chi_1(\kappa,\lambda)^N\,, \qquad n_1 = 1\,, \qquad m_1 = N\,, \tag{3.44}$$

and

$$\sigma(h(\kappa)) \;=\; \{\lambda_{1,1}(\kappa)\}\,, \tag{3.45}$$

$$\lambda_{1,1}(\kappa) \;=\; \sum_{\nu=0}^{\infty} a_\nu^{(1)}\kappa^\nu \quad \text{of multiplicity} \quad m(\lambda_{1,1}(\kappa)) = N\,, \quad a_0^{(1)} = \lambda_0\,, \tag{3.46}$$

$$\Pi_{1,1}(\kappa) \;=\; \mathbb{1} \tag{3.47}$$

$$N_{1,1}(\kappa) \;=\; \sum_{\nu=0}^{\infty} N_\nu^{(1,1)}\,\kappa^\nu \tag{3.48}$$

for all $\kappa \in U_\varepsilon(0)$, some $a_\nu^{(1)} \in \mathbb{C}$ and some $N_\nu^{(1,1)} \in \mathrm{End}(\mathscr{H}_0)$. The eigennilpotent can either be $N_{1,1}(\cdot) \neq 0$ or $N_{1,1}(\cdot) = 0$.

For $t \in \mathbb{C}$ and $\kappa \in U_\varepsilon(0)$ one has

$$e^{-ith(\kappa)} = e^{-it\lambda_{1,1}(\kappa)}\Big(\mathbb{1} + \sum_{k=1}^{N-1} \frac{1}{k!}(-it)^k N_{1,1}(\kappa)^k\Big) . \tag{3.49}$$

**Proof:** (3.45) and (3.46) follow from (3.44) and from Proposition B.3 and Remark B.4 with $r = n_\ell = \ell = j = 1$. Since $\lambda_{1,1}(\cdot)$ is single-valued near zero, by Theorem B.8, 2., the eigenprojection $\Pi_{1,1}(\cdot)$ and the eigennilpotent $N_{1,1}(\cdot)$ are single-valued near zero. This proves (3.47) and (3.48). Proposition B.6 with $r = n_\ell = 1$ and $m_\ell = N$ gives (3.49).

■

**Lemma 3.12** (H1), (H2) and (E3) imply (WLG2), where $\alpha = a_1^{(1)}$ and $\beta = a_2^{(1)}$ are introduced in (3.46).

**Proof:** (H1) and (E3) yield (3.46). In particular, the eigenvalue $\lambda_{1,1}(\cdot)$ of $h(\cdot)$ is of the form $\lambda_{1,1}(\kappa) = \lambda_0 + \alpha\kappa + \beta\kappa^2 + O(|\kappa|^3)$ $(\kappa \to 0)$ for some $\alpha = a_1^{(1)} \in \mathbb{C}$ and $\beta = a_2^{(1)} \in \mathbb{C}$. Then (H2) and Lemma 1.2 imply that $\alpha$ is real.

■

Contrary to (E1) and (E2), in the case (E3) assumptions (H2) and (H3) do not guarantee that $\mathrm{Im}\lambda_{1,1}(\kappa) \neq 0$. However:

**Theorem 3.13** Assume (H1), (H2), (E3) and the notation of Theorem 3.11. If, for $\kappa \in U_\varepsilon(0) \cap (\mathbb{R}\backslash\{0\})$, the eigenvalue $\lambda_{1,1}(\kappa)$ is contained in $\mathbb{C}_- := \{z \in \mathbb{C} \,|\, \mathrm{Im} z < 0\}$, then there exists $\mathbb{N} \ni k_* \geq 2$ such that for all $\kappa \in U_\varepsilon(0) \cap (\mathbb{R}\backslash\{0\})$ one has

$$\mathrm{Im}\lambda_{1,1}(\kappa) = -c\,|\kappa|^{k_*} + O(|\kappa|^{k_*+1}) \quad (\kappa \to 0)\,, \quad c := |\mathrm{Im}\, a_{k_*}^{(1)}| > 0\,. \tag{3.50}$$

In particular, for $\kappa \in U_\varepsilon(0) \cap (\mathbb{R}\backslash\{0\})$,

$$\|e^{-ith(\kappa)}\| \;\leq\; e^{-ct|\kappa|^{k_*}(1+O(|\kappa|))}\Big(1 + \sum_{k=1}^{N-1} O(|t\kappa^2|^k)\Big) \tag{3.51}$$

as $\kappa \to 0$, uniformly for $t \in [0, \infty)$. If $N_{1,1}(\cdot) = 0$, equation (3.49) simplifies to

$$e^{-ith(\kappa)} \;=\; e^{-it\lambda_{1,1}(\kappa)}\mathbb{1} \qquad (\kappa \in U_\varepsilon(0)\,,\; t \in \mathbb{C})\,. \tag{3.52}$$

Then, for $\kappa \in U_\varepsilon(0) \cap (\mathbb{R}\backslash\{0\})$,

$$\|e^{-ith(\kappa)}\| \;\leq\; e^{-tc|\kappa|^{k_*}(1+O(|\kappa|))} \qquad (\kappa \to 0)\,, \tag{3.53}$$

uniformly for $t \in [0, \infty)$.

**Proof:** Assume (H1), (H2) and (E3) and let $\lambda_{1,1}(\kappa) \in \mathbb{C}_-$ for $\kappa \in U_\varepsilon(0) \cap (\mathbb{R}\backslash\{0\})$. By Lemma 3.12, (WLG2) holds with $\alpha = a_1^{(1)}$ and $\beta = a_2^{(1)}$. (WLG2) implies $k_* \geq 2$ and (3.50) follows from (3.46). By Corollary 1.4 one has

$$N_{1,1}(\kappa) := (h(\kappa) - \lambda_{1,1}(\kappa))\Pi_{1,1}(\kappa) = h(\kappa) - \lambda_{1,1}(\kappa) = O(|\kappa|^2) \quad (\kappa \to 0)\,.^{9} \tag{3.54}$$

Inserting (3.50) and (3.54) into (3.49) proves the estimate (3.51). Inserting (3.50) into (3.52) shows (3.53).

■

# A On algebraic functions

## A.1 Polynomials with analytic or meromorphic coefficients

**Definition A.1** For any ring $R$ we denote by $R[\lambda]$ the ring of polynomials in one variable with coefficients in $R$.
For any Riemann surface $X$ let $\mathcal{O}(X)$ denote the ring of analytic functions in $X$ (cf. [Fo, 1.6 Definition]) and let $\mathcal{M}(X)$ denote the field of meromorphic functions in $X$ (cf. [Fo, 1.12 Definition and 1.16 Remark]).
Let $\mathbb{C}\{z - z_0\}$ denote the ring of convergent Taylor series around $z_0 \in \mathbb{C}$ and let $\mathbb{C}\{\{z - z_0\}\}$ be the field of convergent Laurent series around $z_0$ with finite principal part.

**Remark A.2** The association of $U$ open in $X$ to the ring $\mathcal{O}(U)$ defines a sheave $\mathcal{O}$ over $X$, with stalk $\mathcal{O}_{x_0}$ isomorphic to $\mathbb{C}\{z\}$. Unlike $\mathcal{O}(U)$ for $U$ open, $\mathcal{O}_{x_0}$ and $\mathbb{C}\{z\}$ are local rings, i.e., they contain a unique maximal ideal.

Each polynomial $P(\lambda) \in \mathcal{O}(X)[\lambda]$ ($\in \mathcal{M}(X)[\lambda]$, respectively) of degree $n$ has the following representation:

$$P(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \ldots + c_n$$

for some $c_j \in \mathcal{O}(X)$ ($\in \mathcal{M}(X)$, respectively ), $j \in \{0, 1, \ldots, n\}$. Polynomials with $c_0 = 1$ are called normalized.

**Theorem A.3** [Fo, 8.9 Theorem]

Let $X$ be a Riemann surface, and let

$$P(\lambda) = \lambda^n + c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} + \ldots + c_n \in \mathcal{M}(X)[\lambda]$$

be an irreducible (normalized) polynomial of degree $n$. Then there exist a Riemann surface $Y$, a branched *$n$-sheeted holomorphic covering map* $\pi : Y \to X$ and a meromorphic function $F \in \mathcal{M}(Y)$ such that $(\pi^*P)(F) = 0$. ($\pi^*$ denotes the pull back induced by $\pi$; see Remark A.5.) The triple $(Y, \pi, F)$ is uniquely determined modulo a biholomorphic map (see [Fo, 8.9 Theorem] for more details). $(Y, \pi, F)$ is called the algebraic function defined by the polynomial $P$.

For the definition of an *$n$-sheeted holomorphic covering map* $\pi$ we refer to [Fo, 4.24 Remark]. A point $y \in Y$ is called a ramification point (or sometimes a branch point) if there is no neighborhood $V$ of $y$ with $\pi|_V$ injective. (See also Definition A.7 and Remark A.8.)

We remark that the classical instance of Theorem A.3 is for $X = P^1\mathbb{C} := \mathbb{C} \cup \{\infty\}$, the Riemann sphere. Then all coefficients $c_j$ are rational functions. This is the case treated in [Kn]. It is not sufficient for applications to analytic perturbation theory. We follow [Fo] in using also in the case of general $X$ the term *algebraic* function, thus avoiding the term *algebroidal* as used in [B] and [K2].

[9] If, in addition to assumption (E3), all $h^{(\nu)}(0)$ for $\nu \leq k_* - 1$ are diagonalizable, the reduction process of [K2] gives an analog of Lemma 1.2, yielding an improved estimate $O(|\kappa|^{k_*})$ on the r.h.s. of (3.54). In general, this will not hold.

**Definition A.4** [Fo, part of 17.14]

Let $X$ and $Y$ be Riemann surfaces and $f : X \to Y$ a proper (i.e., preimages of compact sets are compact) non-constant holomorphic map. For $x \in X$ let $m(f, x)$ be the multiplicity of $f$ at point $x$ (defined, e.g., in [Fo, 2.2 Remark]). The number

$$b(f, x) := m(f, x) - 1$$

is called the *branching order* of $f$ at point $x$.

**Remark A.5** The pull back induced by $\pi$ is given by

$$\begin{aligned} \pi^* : \mathcal{M}(X) &\to \mathcal{M}(Y) \\ f &\mapsto \pi^* f := f \circ \pi \,; \end{aligned}$$

see [Fo, 8.2]. Thus for $P$, $\pi$, $F$ and $c_j$ ($j \in \{1, \ldots, n\}$) as in Theorem A.3,

$$(\pi^* P)(F) = F^n + (\pi^* c_1)F^{n-1} + (\pi^* c_2)F^{n-2} + \ldots + \pi^* c_n \,.$$

**Corollary A.6** Assume the conditions and notation of Theorem A.3. If $P(\lambda) \in \mathcal{O}(X)[\lambda]$, then $F \in \mathcal{O}(Y)$.

This statement is crucial for applications to analytic perturbation theory. It is implicitly contained in [Fo, 8.14 Remark (1)], without proof.

**Proof of Corollary A.6**: Let

$$P(\lambda) = \lambda^n + c_1 \lambda^{n-1} + \ldots + c_n \in \mathcal{O}(X)[\lambda] \,.$$

Then in particular $P(\lambda) \in \mathcal{M}(X)[\lambda]$. Thus Theorem A.3 yields

$$(\pi^* P)(F) = F^n + (\pi^* c_1)F^{n-1} + \ldots + \pi^* c_n = 0 \tag{A.1}$$

with $F \in \mathcal{M}(Y)$. Since $c_j \in \mathcal{O}(X)$ ($j \in \{1, \ldots, n\}$), we have by [Fo, 1.10 Remark (c)]

$$\begin{aligned} \pi^* : \mathcal{O}(X) &\to \mathcal{O}(Y) \\ c_j &\mapsto \pi^* c_j = c_j \circ \pi \,. \end{aligned}$$

Thus $\pi^* c_j \in \mathcal{O}(Y)$ has no poles ($j \in \{1, \ldots, n\}$). Next we will show $F \in \mathcal{O}(Y)$ by counting pole orders: Assume that there exists a pole $y_0 \in Y$ of $F$, i.e., $F(y_0) = \infty$. Let $\operatorname{ord} F|_{y_0} = -k < 0$, where $\operatorname{ord} F|_{y_0}$ denotes the order of the pole $y_0$ of $F$. (This notation is consistent with the notation in [Fo, 16.2.].) Then

$$F(y) = \sum_{\nu=-k}^{\infty} \alpha_\nu \, (y - y_0)$$

for some $\alpha_\nu \in \mathbb{C}$ with $\alpha_{-k} \neq 0$ and all $y$ in some (sufficiently small) neighborhood of $y_0$. Thus it follows that

$$\operatorname{ord} F^n|_{y_0} = -k \cdot n \,. \tag{A.2}$$

But equation (A.1) is equivalent to

$$F^n = -\sum_{j=1}^{n} (\pi^* c_j) F^{n-j} \,. \tag{A.3}$$

Therefrom it follows that

$$\operatorname{ord} F^n|_{y_0} = \operatorname{ord}\Big( -\sum_{j=1}^{n} (\pi^* c_j) F^{n-j}\Big) = \operatorname{ord}\Big( -\sum_{j=1}^{n-1} (\pi^* c_j) F^{n-j}\Big) \geq -k \cdot (n-1). \quad (A.4)$$

Finally, comparing (A.2) with (A.4) yields

$$-k \cdot n \geq -k \cdot (n-1) \quad \Leftrightarrow \quad -k \geq 0\,,$$

which is a contradiction to $-k < 0$. Thus the meromorphic function $F$ has no poles.

■

**Definition A.7** [Fo, contained in 4.23]

Let $X$, $Y$ be Riemann surfaces. Let $\pi : Y \to X$ be an $n$-sheeted holomorphic covering map (the map $\pi$ is in particular proper non-constant holomorphic). By $\mathcal{A} \subset Y$ we denote the set of all ramification (sometimes branch) points of $\pi$. The set $\mathcal{B} := \pi(\mathcal{A}) \subset X$ is called the set of critical values of $\pi$.

**Remark A.8** Alternatively, one might distinguish between ramification points and branch points by calling $\mathcal{A}$ the set of all ramification points and $\mathcal{B}$ the set of branch points (see [Mi]). This agrees better with the classical texts in function theory.

The sets $\mathcal{A}$ and $\mathcal{B}$ are closed and discrete; see [Fo, 4.23]. By [Fo, 4.24 Theorem] $\pi$ takes every value $x \in X$, counting multiplicities, $n$-times on $Y$; see [Fo, 4.24]. That means

$$n = \sum_{y \in \pi^{-1}(x)} m(\pi, y) \qquad (x \in X)\,,$$

where $m(\pi, y)$ denotes the multiplicity of $\pi$ at point $y$. In particular one has, if # denotes cardinality,

$$\#\big(\pi^{-1}(c)\big) \;=\; n \qquad (c \in X \backslash \mathcal{B})\,, \quad (A.5)$$

$$\#\big(\pi^{-1}(b)\big) \;<\; n \qquad (b \in \mathcal{B})\,. \quad (A.6)$$

We shall now determine the local structure of an algebraic function near a critical value, first algebraically (in Appendix A.1.1) using the local ring $\mathbb{C}\{z\}$ and then by more topological arguments (in Appendix A.1.2).

### A.1.1 Puiseux representation

Let $b \in \mathcal{B}$ be a critical value of the branched $n$-sheeted holomorphic covering map $\pi : Y \to X$ associated with a normalized $P(\lambda) \in \mathcal{O}(X)[\lambda]$ (by Theorem A.3 and Corollary A.6). Using a holomorphic chart centered[10] at $b$, $P(\lambda)$ induces a polynomial $\widetilde{P}(\lambda) \in \mathbb{C}\{z\}[\lambda]$, which in general is irreducible:

Any normalized polynomial $\widetilde{P}(\lambda) \in \mathbb{C}\{z\}[\lambda]$ of degree $n$ uniquely factorizes into a product of irreducible polynomials, i.e.,

$$\widetilde{P}(\lambda) = \prod_{\ell=1}^{r} P_\ell(\lambda)^{m_\ell} \quad (A.7)$$

for some $m_\ell \in \mathbb{N}$ and some $r \leq n$ with $P_\ell(\lambda) \in \mathbb{C}\{z\}[\lambda]$ irreducible and normalized, $P_\ell(\lambda) \neq P_{\ell'}(\lambda)$ $(\ell \neq \ell')$, $\ell, \ell' \in \{1, \ldots, r\}$. If $n_\ell$ denotes the degree of $P_\ell(\lambda)$, one has $\sum_{\ell=1}^{r} m_\ell \cdot n_\ell = n$. The unique factorization in (A.7) is proven by the following two theorems:

[10] A chart $\psi$ is called centered at a point $x_0$, if $\psi(x_0) = 0$.

**Theorem A.9** [F, Kapitel 6.11, p.89, Theorem]

> The ring $\mathbb{C}\{z\}$ of convergent Taylor series is factorial.

Here we follow the notation of [L]: A unique factorization ring or a unique factorization domain or factorial ring denote the same thing.

The proof of Theorem A.9 uses the Weierstraß Preparation Theorem. We will not reproduce the proof here. We refer the reader to, e.g., [F], [BrKnö, Chapter 8.2, Proposition 6, p.347] or [Gu, Chapter A], [GH, Chapter 5.3, p.678 ff. and Chapter 0, p.8 ff.].

**Theorem A.10** [L, Chapter IV §2, Theorem 2.3]

> Let $R$ be a factorial ring and $R[\lambda]$ the ring of polynomials in one variable with coefficients in $R$. Then $R[\lambda]$ is factorial. Let $K$ denote the quotient field of $R$. The prime elements of $R[\lambda]$ are the primes of $R$ and polynomials in $R[\lambda]$ which are irreducible in $K[\lambda]$ and have content 1.

For a precise definition of content 1 we refer to [L, IV §2]. Roughly speaking, a polynomial $p(\lambda) = \sum_{j=1}^{n} c_j \lambda^j \in R[\lambda]$ has content 1 if the greatest common divisor of the $c_j$ ($j \in \{1, \ldots, n\}$) is 1. In particular, normalized polynomials always have content 1. For polynomials $p(\lambda) \in R[\lambda]$ with content 1 irreducibility in $R[\lambda]$ and $K[\lambda]$ coincide.

We shall now formulate a local version of Theorem A.3, using more classical terms. In particular, we shall implicitly identify a polynomial $P(\lambda)$ in $\mathbb{C}\{\{z\}\}[\lambda]$ with the associated map $(z, \lambda) \mapsto P(z, \lambda)$.

**Theorem A.11 (Puiseux)** [Fo, 8.14 Theorem and Remark (1)]

> Let
>
> $$P(z, \lambda) = \lambda^n + c_1(z)\lambda^{n-1} + \ldots + c_n(z) \in \mathbb{C}\{\{z\}\}[\lambda]$$
>
> be an irreducible normalized polynomial of degree $n$ over the field $\mathbb{C}\{\{z\}\}$. Then there exist $k \in \mathbb{Z}$ and a Laurent series
>
> $$\phi(\zeta) = \sum_{\nu=k}^{\infty} a_\nu \zeta^\nu \in \mathbb{C}\{\{\zeta\}\}$$
>
> such that
>
> $$P(\zeta^n, \phi(\zeta)) = 0$$
>
> as an element of $\mathbb{C}\{\{\zeta\}\}$.
> Furthermore, if $c_j \in \mathbb{C}\{z\}$ ($j \in \{1, \ldots, n\}$), then $\phi(\zeta) \in \mathbb{C}\{\zeta\}$.

An even more classical formulation of Theorem A.11 is:

**Remark A.12** [Fo, 8.14 Remark (2)]

> The equation
>
> $$\mathbb{C}\{\{z\}\}[\lambda] \ni P(z, \lambda) = 0$$

can be solved by a Puiseux-Laurent series

$$\lambda = \phi(\sqrt[n]{z}) = \sum_{\nu=k}^{\infty} a_\nu z^{\nu/n}$$

with some $k \in \mathbb{Z}$. Furthermore,

$$\mathbb{C}\{z\}[\lambda] \ni P(z,\lambda) = 0$$

can be solved by a Puiseux series

$$\lambda = \phi(\sqrt[n]{z}) = \sum_{\nu=0}^{\infty} a_\nu z^{\nu/n}\,. \tag{A.8}$$

### A.1.2 Local decomposition into connected components

**A.1.2.1 Decomposition over neighborhoods of critical values:** We shall now decompose (by topological arguments) $\pi : Y \to X$ over a small open neighborhood $U(b)$ of a critical value (or branch point; cf. Definition A.7 and Remark A.8) $b \in \mathcal{B}$:

Let $b \in \mathcal{B}$ be a critical value of the branched $n$-sheeted holomorphic covering map $\pi : Y \to X$, associated to $P(\lambda) \in \mathcal{O}(X)[\lambda]$ normalized and irreducible (by Theorem A.3 and Corollary A.6). Let $U(b)$ be an open sufficiently small neighborhood of $b$ which contains no other critical value of $\pi$ than $b$. Then, since $\pi$ is an $n$-sheeted covering map, the preimage of $U(b)$ under $\pi$ is a disjoint union of open connected subsets $B_\ell \subset Y$,

$$\pi^{-1}\big(U(b)\big) = \bigcup_{\ell=1}^{r} B_\ell$$

for some $r < n$. (This holds, since the preimage of an open connected set is open, and every open set decomposes into connected components.)

Choosing a holomorphic chart $\psi$, centered at $b \in X$, and possibly shrinking $U(b)$, we may assume that $U(b) = \psi^{-1}(U_1(0))$, where $U_1(0)$ is the open unit disc in $\mathbb{C}$. Define the slit neighborhood of $b$,

$$D(b) := \psi^{-1}(D_1(0))\,, \quad D_1(0) := U_1(0) \backslash (-\infty, 0]\,. \tag{A.9}$$

Then $\pi^{-1}\big(D(b)\big) \cap B_\ell$ decomposes into $n_\ell$ connected components $C_{\ell,j}$:

$$\pi^{-1}\big(D(b)\big) \cap B_\ell = \bigcup_{j=1}^{n_\ell} C_{\ell,j}\,. \tag{A.10}$$

We shall prove (A.10) by use of the following theorem.

**Theorem A.13** [Fo, 5.11 Theorem]

Let $Y$ be a Riemann surface. Let $U_1(0)$ be the open unit disk around 0 and $f : Y \to U_1(0)$ a proper non-constant holomorphic map, $f$ unbranched over $U_1(0)\backslash\{0\}$. Then there exist a biholomorphic map $\phi : Y \to U_1(0)$ and $k \in \mathbb{N}\backslash\{0\}$ such that

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & U_1(0) \arrow[dl, "p_k"] \\ & U_1(0) & \end{tikzcd}$$

is a commutative diagram, where $p_k(z) := z^k$.

**Proof of decomposition (A.10)**: Note that $\pi^{-1}(D(b)) \cap B_\ell$ is a Riemann surface and

$$\psi \circ \pi_\ell := \pi \restriction \pi^{-1}(D(b)) \cap B_\ell \to U_1(0)$$

is unbranched by construction (since the branch point $\pi^{-1}(b)$ has been removed) and trivially a proper non-constant holomorphic map (since $\pi$ is a covering). Then applying Theorem A.13 gives a biholomorphic map $\phi_\ell : \pi^{-1}(D(b)) \cap B_\ell \to U_1(0)$ and an $n_\ell \in \mathbb{N}\backslash\{0\}$ such that, for $p_{n_\ell}(z) := z^{n_\ell}$, the diagram

$$\begin{tikzcd}
\pi^{-1}(D(b)) \cap B_\ell \arrow[rr, "\phi_\ell"] \arrow[dr, "\psi \circ \pi_\ell"'] & & U_1(0) \arrow[dl, "p_{n_\ell}"] \\
& U_1(0) &
\end{tikzcd}$$

is commutative. We thus have $\psi \circ \pi_\ell = p_{n_\ell} \circ \phi_\ell = \phi_\ell^{n_\ell}$.

■

**Remark A.14** Roughly speaking, each $B_\ell$ consists of $n_\ell$ sheets which meet in $\pi^{-1}(b) \cap B_\ell$. The number $n_\ell$ coincides with the degree of the polynomial $P_\ell$ introduced in (A.7). This can be seen as follows: For a sufficiently small neighborhood $U$ of $b \in \mathcal{B}$, the coefficients of $\widetilde{P}(\lambda)$ and $P_\ell(\lambda)$ actually converge in $U$. Thus $\widetilde{P}(\lambda)$ and $P_\ell(\lambda)$ may be considered as polynomials in $\mathcal{O}(U)[\lambda]$, and a fortiori (A.7) is a decomposition into irreducible elements in $\mathcal{O}(U)[\lambda]$. For each $P_\ell(\lambda)$ Theorem A.3 gives a (modulo biholomorphic maps) unique algebraic function $(Y_\ell, \pi_\ell, F_\ell)$ over $U$. Since the $Y_\ell$ are connected by definition, they necessarily must give the connected components of $\pi^{-1}(U)$ for the original algebraic function $(Y, \pi, F)$.

Now

$$\pi_{\ell,j} := \pi \restriction C_{\ell,j} \to D(b) \qquad (j \in \{1, \ldots, n_\ell\})$$

defines an unbranched (by construction) and biholomorphic map (by Theorem A.13). Then

$$\lambda_{\ell,j}(z) := F|_{C_{\ell,j}}\big((\psi \circ \pi_{\ell,j})^{-1}(z)\big) = \big(F|_{C_{\ell,j}} \circ \pi_{\ell,j}^{-1} \circ \psi^{-1}\big)(z) \qquad (z \in \psi(D(b))) \tag{A.11}$$

is an analytic function of $z \in \psi(D(b)) = D_1(0)$.

**A.1.2.2 Decomposition over neighborhoods without critical values:** We shall now decompose $\pi : Y \to X$ over a small open neighborhood of a point $x_0 \in X\backslash\mathcal{B}$, i.e., $x_0$ is not a critical value of $\pi$, where $\pi$ is associated to $P(\lambda) \in \mathcal{O}(X)[\lambda]$ irreducible and normalized. Let $U(x_0)$ be a sufficiently small open neighborhood of $x_0$ which contains no critical values of $\pi$. In analogy to (A.9), choosing a holomorphic chart $\psi$ centered at $x_0$ and possibly shrinking $U(x_0)$, we may assume that $U(x_0) = \psi^{-1}(U_1(0))$, where $U_1(0)$ is the open unit disc in $\mathbb{C}$. Then, since $\pi$ is an $n$-sheeted covering, the preimage of $U(x_0)$ under $\pi$ is a disjoint union of $r$ open connected subsets $B_\ell \subset Y$,

$$\pi^{-1}\big(U(x_0)\big) = \bigcup_{\ell=1}^{r} B_\ell$$

for some $r \leq n$. Then

$$\pi_\ell := \pi \restriction \pi^{-1}\big(U(x_0)\big) \cap B_\ell \to U(x_0)$$

is unbranched (since $\pi^{-1}\big(U(x_0)\big)$ contains no branch points of $\pi$) and trivially a non-constant holomorphic map (since $\pi$ is a covering). Then

$$\lambda_\ell(z) := F|_{B_\ell}\big((\psi\circ\pi_\ell)^{-1}(z)\big) = \big(F|_{B_\ell}\circ\pi_\ell^{-1}\circ\psi^{-1}\big)(z) \qquad (z\in\psi(U(x_0)) \tag{A.12}$$

is an analytic function of $z\in\psi(U(x_0)) = U_1(0)$. To stay consistent with the notation in (A.11) and (A.15) we will then write

$$\lambda_{\ell,1}(z) := F|_{B_{\ell,1}}\big((\psi\circ\pi_{\ell,1})^{-1}(z)\big) = \big(F|_{B_{\ell,1}}\circ\pi_{\ell,1}^{-1}\circ\psi^{-1}\big)(z) \quad (z\in\psi(U(x_0)) \tag{A.13}$$

instead of (A.12).

**Remark A.15** In analogy to Remark A.14 we then say that $B_\ell$ consists of just $n_\ell = 1$ sheet.

## A.2 Classical language: multi-valued analytic functions

Using the same setting and notation as in Appendix A.1.2.1, in classical language each index $\ell\in\{1,\dots,r\}$ describes "a multi-valued analytic function (multi $= n_\ell$)" and each $\lambda_{\ell,j}(z)$, $j\in\{1,\dots,n_\ell\}$, is "a branch of this $n_\ell$-valued analytic function". In the language of [K2, II §1.2] (following the reference [Kn]), the functions $\{\lambda_{\ell,j}(z)\}_{j=1}^{n_\ell}$, $z\in\psi(D(b))$, form a *cycle* at the *exceptional* [11] point $b$ (since they transform one into another under analytic continuation around $b$), and the number $n_\ell$ of sheets of $B_\ell$ is called the *period* of this cycle. Therefore $\{\lambda_{\ell,j}(z)\}_{j=1}^{n_\ell}$, $z\in\psi(D(b))$, is also called an $n_\ell$-cycle (at point $b$).

For $z\in\psi(D(b))$ the function $\lambda_{\ell,j}(z)$ is a solution of $P_\ell(z,\lambda)=0$, where $P_\ell(z,\lambda)$ is an irreducible polynomial in the unique factorization (A.7). Thus by Theorem A.11, the function $\lambda_{\ell,j}(z)$ $(z\in\psi(D(b))$ is a convergent Puiseux series in the variable

$$\sqrt[n_\ell]{z}\,, \tag{A.14}$$

where (A.14) should be understood as the $j$th branch of $\sqrt[n_\ell]{\cdot}$. Thus according to (A.8) we write

$$\lambda_{\ell,j}(z) = \phi_{\ell,j}(\sqrt[n_\ell]{z}) := \sum_{\nu=0}^{\infty} a_\nu^{(\ell,j)} z^{\nu/n_\ell}\,, \tag{A.15}$$

$$a_\nu^{(\ell,j)} := a_\nu^{(\ell)}\omega^{\nu(j-1)}\,,\quad \omega := e^{2\pi i/n_\ell} \tag{A.16}$$

for some $a_\nu^{(\ell)}\in\mathbb{C}$ and all $z\in\psi(D(b))$.

We emphasize that in this description we freely use equivalence of the algebraic and topological local decomposition according to Section A.1.1 and Section A.1.2.

For the sake of completeness we finally shall describe the special case corresponding to the situation of Appendix A.1.2.2.

**Single-valued analytic functions:** Assume the same setting and notation as in Appendix A.1.2.2. Then for $z\in\psi(U(x_0))$ the function $\lambda_{\ell,1}(z)$ is a solution of $P_\ell(z,\lambda)=0$ $(\ell\in\{1,\dots,r\})$. (The polynomials $P_\ell(z,\lambda)\in\mathbb{C}\{z\}[\lambda]$ have been introduced in (A.7).) By Theorem A.11 one gets

$$\lambda_{\ell,1}(z) = \phi_{\ell,1}(z) := \sum_{\nu=0}^{\infty} a_\nu^{(\ell,1)} z^\nu \qquad (\ell\in\{1,\dots,r\}) \tag{A.17}$$

for some $a_\nu^{(\ell,1)}\in\mathbb{C}$ and all $z\in\psi(U(x_0))$.

**Remark A.16** Following the classical language described at the beginning of this Appendix A.2, we call $\lambda_{\ell,1}(z)$ $(z\in\psi(U(x_0)))$ a 1-cycle at point $x_0$.

[11] See Remark B.7 for a definition and more details.

# B Jordan decomposition and characteristic polynomial of the analytic matrix family $h(\kappa)$

## B.1 General facts

Since $\mathcal{M}(X)$ is a field for any Riemann surface $X$ (see, e.g., [Fo, 1.16 Remark]), it is in particular a factorial ring. Thus, by Theorem A.10, the ring $\mathcal{M}(X)[\lambda]$ is factorial. Then for any normalized polynomial $P(\lambda) \in \mathcal{M}(X)[\lambda]$ there exists a unique factorization

$$P(\lambda) = \prod_{\ell=1}^{r} P_\ell(\lambda)^{m_\ell}\,, \qquad \text{degree}(P(\lambda)) = \sum_{\ell=1}^{r} m_\ell \cdot \text{degree}(P_\ell(\lambda)) \tag{B.1}$$

for some $r \le \text{degree}(P(\lambda))$ and some $m_\ell \in \mathbb{N}$, $P_\ell(\lambda) \in \mathcal{M}(X)[\lambda]$ normalized and irreducible, $P_\ell(\lambda) \neq P_{\ell'}(\lambda)$, $\ell \neq \ell'$, $\ell, \ell' \in \{1, \ldots, r\}$. We even have

**Theorem B.1** Let $X$ be a non-compact Riemann surface, $P(\lambda) \in \mathcal{O}(X)[\lambda]$. Then the functions $P_\ell(\lambda)$ in the unique factorization (B.1), which a priori are in $\mathcal{M}(X)[\lambda]$ actually are in $\mathcal{O}(X)[\lambda]$.

We emphasize that Theorem B.1 is not an immediate consequence of the standard result Theorem A.10, since no ring $\mathcal{O}(X)$ is factorial (see [Rem, Chapter 4* §2, p.94]), for $X$ a non-compact Riemann surface. Theorem B.1 crucially depends on the fact that the subring $\mathcal{O}(X)$ of $\mathcal{M}(X)$ is actually defined by local conditions.[12]

**Proof of Theorem B.1**: Arguing inductively, it suffices to show

($\star$) If $P_j(\lambda) = \sum_{i=0}^{n_j} a_{ij}\lambda^{n_j - i} \in \mathcal{M}(X)[\lambda]$ ($j \in \{1,2\}$) are normalized,
if $P_1(\lambda)P_2(\lambda) = P(\lambda) = \sum_{i=0}^{n} b_i\lambda^{n-i}$ and $A_j \subset X$ denotes the set of poles of the coefficients $a_{ij}$ of $P_j(\lambda)$, then any $z_0 \in A_1 \cup A_2$ is a pole of (at least one of) the coefficients $b_i$ of $P(\lambda)$.

Assume to the contrary of ($\star$) that there is $z_0 \in A_1 \cup A_2$ with all $b_i$ analytic at $z_0$. Choosing a chart $\psi$ centered at $z_0$ and denoting by $\widetilde{P}(\lambda), \widetilde{P}_j(\lambda) \in \mathbb{C}\{\{z\}\}[\lambda]$ the polynomials induced from $P(\lambda), P_j(\lambda)$ and $\psi$ we have

$$\widetilde{P}(\lambda) = \widetilde{P}_1(\lambda)\widetilde{P}_2(\lambda) \tag{B.2}$$

with $\widetilde{P}(\lambda) \in \mathbb{C}\{z\}[\lambda]$. Since $\mathbb{C}\{z\}$ is factorial, we may apply Theorem A.10 to obtain a unique factorization into prime elements of the l.h.s. and the r.h.s. of (B.2) which all belong to $\mathbb{C}\{z\}[\lambda]$. But this implies $\widetilde{P}_1(\lambda), \widetilde{P}_2(\lambda) \in \mathbb{C}\{z\}[\lambda]$, contradicting $z_0 \in A_1 \cup A_2$.

■

We remark that we have taken the reduction to ($\star$) from Baumgärtel's book [B, proof of Theorem 1, Anhang §2.6 ], where the above theorem is stated and proven for $X$ an open connected subset of $\mathbb{C}$, using a more analytic argument (i.e., performing the limit $z \to z_0$).

[12] At this point it obviously is a real disaster to base analytic perturbation theory on [Kn]. Then $X$ is the Riemann sphere, thus compact, by Liouville $\mathcal{O}(X)$ is trivial (equal to $\mathbb{C}$), the above theorem is trivially true (better: void), and there is very probably not a single meaningful application of the resulting perturbation theory to problems of quantum mechanics. It is the great merit of [B] that this has been remedied. Very implicitly this is correctly acknowledged in the footnote on p.64 in [K2]. To understand this state of affairs, the uninitiated reader needs careful comparative study of [Kn], [K2] and [B], preferably in the light of modern references. This is not what one would like to recommend to a newcomer to the field. Textbook presentation needs an update.

We shall now apply these results to the characteristic polynomial of an analytic matrix family $h(\kappa)$.
Let $X \subset \mathbb{C}$ be an open connected subset (which in particular is a Riemann surface). Let $\mathscr{H}_0$ denote a complex vector space with $\dim \mathscr{H}_0 =: N < \infty$. Let $h(\kappa)$, $\kappa \in X$, denote an analytic matrix family acting on $\mathscr{H}_0$. Then

$$\chi(\lambda) \quad := \quad \det(h(\cdot) - \lambda) \in \mathcal{O}(X)[\lambda] \subset \mathcal{M}(X)[\lambda] \tag{B.3}$$

denotes the characteristic polynomial of $h(\cdot)$. We identify $\chi(\lambda)$ with the associated map,

$$\begin{aligned} \chi : \ X \times \mathbb{C} \quad &\to \quad P^1\mathbb{C} \\ (\kappa, \lambda) \quad &\mapsto \quad \chi(\kappa, \lambda) := \det(h(\kappa) - \lambda)\,, \end{aligned} \tag{B.4}$$

where $P^1\mathbb{C} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. For analyzing

$$\chi(\kappa, \lambda) = 0\,, \tag{B.5}$$

one first factorizes $\chi(\lambda)$ into irreducible factors according to Theorem B.1:

$$\chi(\lambda) = \prod_{\ell=1}^{r} \chi_\ell(\lambda)^{m_\ell} \tag{B.6}$$

for some $r \leq N$ and some $m_\ell \in \mathbb{N}$ with $\chi_\ell(\lambda) \in \mathcal{O}(X)[\lambda]$ normalized and irreducible, $\chi_\ell(\lambda) \neq \chi_{\ell'}(\lambda)$, $\ell \neq \ell'$, $\quad \ell, \ell' \in \{1, \ldots, r\}$. Then

$$\sum_{\ell=1}^{r} m_\ell \cdot n_\ell = N\,, \qquad\qquad n_\ell := \mathrm{degree}(\chi_\ell(\lambda))\,. \tag{B.7}$$

Applying Theorem A.3 together with Corollary A.6 to each of the $\chi_\ell(\lambda)$ ($\ell \in \{1, \ldots, r\}$) in the unique factorization (B.6) yields that there exist a Riemann surface $Y_\ell$, a branched $n_\ell$-sheeted holomorphic covering map $\pi_\ell : Y_\ell \to X$ and an analytic function $F_\ell \in \mathcal{O}(Y_\ell)$, such that $(\pi_\ell^* \chi_\ell)(F_\ell) = 0$ ($\ell \in \{1, \ldots, r\}$). The algebraic function $(Y_\ell, \pi_\ell, F_\ell)$ defined by $\chi_\ell$ is uniquely determined ($\ell \in \{1, \ldots, r\}$), modulo a biholomorphic map.

Thus combining Theorem B.1 with Theorem A.3 proves that, in the language of [K2, II §1.1, p.64], the eigenvalues of $h(\kappa)$ are branches of one or more analytic functions with only algebraic singularities. Here Theorem B.1 and Theorem A.3 replace the "well-known result in function theory" from Knopp's 1920 book [Kn].

As in Appendix A.1.1 and Appendix A.1.2, we shall now analyze the algebraic function $(Y_\ell, \pi_\ell, F_\ell)$ locally near $x_0 \in X$. Using a chart centered at $x_0 \in X$, giving an isomorphism $\mathcal{O}_{x_0} \simeq \mathbb{C}\{z\}$, we obtain a map

$$\mathcal{O}(X)[\lambda] \ni \chi_\ell(\lambda) \longmapsto \widetilde{\chi}_\ell(\lambda) = \prod_{\rho}^{s} \widetilde{\chi}_{\ell,\rho}(\lambda)^{m_\rho} \in \mathbb{C}\{z\}[\lambda]\,, \tag{B.8}$$

$$\sum_{\rho=1}^{s} m_\rho \cdot n_\rho = n_\ell\,, \qquad n_\rho := \mathrm{degree}(\chi_{\ell,\rho}(\lambda))$$

for some $s \leq r$, where $\widetilde{\chi}_{\ell,\rho}(\lambda) \in \mathbb{C}\{z\}[\lambda]$ is normalized and irreducible, $\widetilde{\chi}_{\ell,\rho}(\lambda) \neq \widetilde{\chi}_{\ell,\rho'}(\lambda)$, $\rho \neq \rho'$, $\quad \rho, \rho' \in \{1, \ldots, s\}$. Thus we may assume without loss of generality that $X$ is a sufficiently small open disc around zero,

$$X = U_\varepsilon(0) := \{\kappa \in \mathbb{C} \,|\, |\kappa - 0| < \varepsilon\} \quad \text{for some } \varepsilon > 0 \text{ sufficiently small,} \tag{B.9}$$

and the irreducible factorization (B.6) in $\mathcal{O}(X)[\lambda]$ coincides with an irreducible factorization in $\mathbb{C}\{z\}[\lambda]$,

$$\chi(\lambda) = \prod_{\ell=1}^{r} \chi_\ell(\lambda)^{m_\ell}\,, \qquad \sum_{\ell=1}^{r} m_\ell \cdot n_\ell = N, \tag{B.10}$$

where $\chi_\ell(\lambda)$ is both in $\mathbb{C}\{z\}[\lambda]$ and $\mathcal{O}(X)[\lambda]$. By Theorem A.11 solutions of

$$\chi_\ell(\kappa,\lambda) = 0 \tag{B.11}$$

are given by convergent Puiseux series

$$\lambda_{\ell,j}(\kappa) = \sum_{\nu=0}^{\infty} a_\nu^{(\ell,j)} \kappa^{\nu/n_\ell} \qquad (\ell \in \{1,\dots,r\},\ j \in \{1,\dots,n_\ell\}) \tag{B.12}$$

for some $a_\nu^{(\ell,j)} \in \mathbb{C}$ and all $\kappa$ in the slit open disc

$$D_\varepsilon(0) := U_\varepsilon(0)\backslash(-\infty,0]\,. \tag{B.13}$$

Obviously, solutions (B.12) of (B.11) are solutions of (B.5). Using the previous results and notation of this Appendix B, we summarize:

**Lemma B.2** Let $U_\varepsilon(0)$ be given by (B.9). Let $h(\kappa)$, $\kappa \in U_\varepsilon(0)$, be an analytic matrix family. Then, for $\varepsilon > 0$ sufficiently small, the characteristic polynomial $\chi(\kappa,\lambda)$ of $h(\kappa)$ has a unique factorization

$$\chi(\kappa,\lambda) = \prod_{\ell=1}^{r} \chi_\ell(\kappa,\lambda)^{m_\ell}\,, \tag{B.14}$$

irreducible both in $\mathbb{C}\{\kappa\}[\lambda]$ and $\mathcal{O}(U_\varepsilon(0))[\lambda]$. Here the polynomials $\chi_\ell(\kappa,\lambda)$ have degree $n_\ell$ and are pairwise distinct ($\ell \in \{1,\dots,r\}$) with $\sum_{\ell=1}^{r} m_\ell \cdot n_\ell = N$.

The Puiseux expansion of Appendix A.1.1, summarized before (A.15), gives

**Proposition B.3** Assume (H1). All eigenvalues of $h(\kappa)$ for $\kappa$ near zero (i.e., the $\lambda_0$-group for $h(\kappa)$ in the sense of Kato; cf. [K2, Chapter II §1.2]) are given by convergent Puiseux series

$$\lambda_{\ell,j}(\kappa) = \sum_{\nu=0}^{\infty} a_\nu^{(\ell,j)} \kappa^{\nu/n_\ell}\,, \qquad a_\nu^{(\ell,j)} := a_\nu^{(\ell)} \omega^{\nu(j-1)}\,, \qquad \omega := e^{2\pi i/n_\ell} \tag{B.15}$$

for some $a_\nu^{(\ell)} \in \mathbb{C}$ and all $\kappa \in D_\varepsilon(0)$, $\ell \in \{1,\dots,r\}$, $j \in \{1,\dots,n_\ell\}$; cf. [K2, II §1.2, (1.7)].

**Remark B.4** In classical language each index $\ell$ describes an $n_\ell$-valued analytic function (corresponding to the polynomial $\chi_\ell$ in the unique factorization (B.10) (or (B.14))), and each eigenvalue $\lambda_{\ell,j}(\kappa)$ is a branch of this $n_\ell$-valued analytic function. For fixed $\ell$ the functions $\{\lambda_{\ell,j}(\kappa)\}_{j=1}^{n_\ell}$, $\kappa \in D_\varepsilon(0)$, form the $\ell$th cycle at point $\kappa = 0$. (Cf. Appendix A.2 and Remark A.16.) All eigenvalues $\{\lambda_{\ell,j}(\kappa)\}_{j=1}^{n_\ell}$, $\kappa \in D_\varepsilon(0)$, of the $\ell$th cycle have the same algebraic multiplicity

$$m(\lambda_{\ell,j}(\cdot)) = m_\ell\,. \tag{B.16}$$

According to (B.15), $\lambda_{\ell,1}(\cdot)$ corresponds to the principal branch of $\sqrt[n_\ell]{\cdot}$. Under analytic continuation around 0, the branches $\lambda_{\ell,j}$ transform one into another as follows:

$$\lambda_{\ell,1} \to \lambda_{\ell,2} \to \lambda_{\ell,3} \to \dots \to \lambda_{\ell,n_\ell-1} \to \lambda_{\ell,n_\ell} \to \lambda_{\ell,n_\ell+1} = \lambda_{\ell,1}\,.$$

Note that $\lambda_{n_\ell+1} = \lambda_1$, since $\omega^{n_\ell} = e^{2\pi i} = 1 = \omega^0$.

**Definition B.5** Let $\mathscr{H}_0$ be a complex vector space with $\dim\mathscr{H}_0 =: N < \infty$. Let $X = U_\varepsilon(0)$ and $D_\varepsilon(0)$ be given by (B.9) and (B.13) for some $\varepsilon > 0$ chosen sufficiently small. Let $h(\kappa)$ ($\kappa \in U_\varepsilon(0)$) denote a family of analytic endomorphisms on $\mathscr{H}_0$. Assume the factorization (B.10) (or (B.14)) of the characteristic polynomial $\chi(\lambda) \in \mathbb{C}\{\kappa\}[\lambda]$ of $h(\cdot)$. Let $\lambda_{\ell,j}(\cdot)$ denote the $\sum_{\ell=1}^r n_\ell$ pairwise distinct eigenvalues of $h(\cdot)$ with algebraic multiplicities $m(\lambda_{\ell,j}(\cdot)) = m_\ell$. Then

1. For $\kappa \in D_\varepsilon(0)$, $\ell \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n_\ell\}$ the eigenprojections corresponding to $\lambda_{\ell,j}(\kappa)$ are the Riesz projections
$$\Pi_{\ell,j}(\kappa) := -\frac{1}{2\pi i} \oint_{\Gamma_{\ell,j}(\kappa)} (h(\kappa) - z)^{-1}\, dz\,, \tag{B.17}$$
where $\Gamma_{\ell,j}(\kappa)$ is some curve in the resolvent set of $h(\kappa)$, enclosing $\lambda_{\ell,j}(\kappa)$ but no other eigenvalue of $h(\kappa)$ different from $\lambda_{\ell,j}(\kappa)$.
2. For $\kappa \in D_\varepsilon(0)$, $\ell \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n_\ell\}$ the eigennilpotents corresponding to $\lambda_{\ell,j}(\kappa)$ are given by
$$N_{\ell,j}(\kappa) := (h(\kappa) - \lambda_{\ell,j}(\kappa))\Pi_{\ell,j}(\kappa)\,. \tag{B.18}$$

Based on the Jordan decomposition of $h(\kappa)$,
$$h(\kappa) = \sum_{\ell,j} \lambda_{\ell,j}(\kappa)\Pi_{\ell,j}(\kappa) + N_{\ell,j}(\kappa), \tag{B.19}$$
the following proposition (which actually is a special case of the results in [K2, I §5.6, (5.50) and (5.51]) precisely shows the contributions of the eigenvalues, the eigenprojections and eigennilpotents of $h(\kappa)$ to the dynamics $e^{-ith(\kappa)}$:

**Proposition B.6** Under the assumptions and notation of Definition B.5, for $\kappa \in D_\varepsilon(0)$ and $t \in \mathbb{C}$, one has
$$e^{-ith(\kappa)} = \sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell} e^{-it\lambda_{\ell,j}(\kappa)}\Pi_{\ell,j}(\kappa) + N_{\ell,j}(\kappa,t)'\,, \tag{B.20}$$
where
$$N_{\ell,j}(\kappa,t)' := \sum_{k=1}^{m_\ell - 1} \frac{1}{k!}\,(-it)^k\, e^{-it\lambda_{\ell,j}(\kappa)}\, N_{\ell,j}(\kappa)^k \qquad (m_\ell \geq 2)\,, \tag{B.21}$$
$$N_{\ell,j}(\kappa,t)' := 0 \qquad (m_\ell = 1)\,. \tag{B.22}$$

**Proof**: Using $\sum_{\ell,j} \Pi_{\ell,j}(\kappa) = \mathbb{1}$ , we write
$$e^{-ith(\kappa)} = \sum_{\ell,j} e^{-ith(\kappa)\Pi_{\ell,j}(\kappa)}\Pi_{\ell,j}(\kappa)\,. \tag{B.23}$$

Inserting $h(\kappa)\Pi_{\ell,j}(\kappa) \overset{(B.19)}{=} \sum_{\ell,j} \lambda_{\ell,j}(\kappa)\Pi_{\ell,j}(\kappa) + N_{\ell,j}(\kappa)$ into the r.h.s. of (B.23) and using (for $m_\ell \geq 2$)
$$e^{-itN_{\ell,j}(\kappa)} = \mathbb{1} + \sum_{k=1}^{m_\ell - 1} \frac{(-it)^k}{k!} \left(N_{\ell,j}(\kappa)\right)^k$$
gives (B.21).

■

**Remark B.7** Originally, in [K2, II §1.1 and §1.2 Remark 1.3] an exceptional point $\kappa_0$ is defined to be a point in $X$ ($X$ some open connected subset of $\mathbb{C}$), where there is a splitting of $\lambda_0$ ($\lambda_0$ an eigenvalue of $h(\kappa_0)$) under perturbation. "Thus there is always splitting at (and only at) an exceptional point". More precisely:

For $X$ a Riemann surface, $h(\kappa)$ analytic in $\kappa \in X$, the characteristic polynomial has a factorization

$$\chi(\lambda) = \prod_{\ell=1}^{r} \chi_\ell(\lambda)^{m_\ell}\,, \tag{B.24}$$

irreducible in $\mathcal{O}(X)[\lambda]$, where by Theorem A.3 $\chi_\ell(\lambda)$ is associated with an algebraic function $(Y_\ell, \pi_\ell, F_\ell)$. Then the point $\kappa_0 \in X$ is exceptional, if either $\kappa_0$ is a critical value of $\pi_\ell$ for at least one $\ell \in \{1, \ldots, r\}$, or if there is accidental degeneracy in the following sense: In the factorization (B.24) there are at least two distinct irreducible polynomials $\chi_\ell(\lambda) \neq \chi_{\ell'}(\lambda)$, such that the associated algebraic functions $(Y_\ell, \pi_\ell, F_\ell)$ and $(Y_{\ell'}, \pi_{\ell'}, F_{\ell'})$ coincide somewhere in the fibre over $\kappa_0$, i.e. $F_\ell(y) = F_{\ell'}(y')$ for some points with $\pi_\ell(y) = \kappa_0 = \pi_{\ell'}(y')$.

Knowing the form of eigenvalues allows to draw conclusions about the form of the corresponding eigenprojections and eigennilpotents. Results in this direction are formulated in the following theorems, freely quoted from [K2]:

**Theorem B.8** [K2, II §1.5 Theorem 1.8]

Let $X$ be an open connected subset of $\mathbb{C}$. Let $\mathscr{H}_0$ be a finite-dimensional complex vector space and let $h(\cdot) : X \to \mathrm{End}(\mathscr{H}_0)$, $\kappa \mapsto h(\kappa)$ denote an analytic matrix family. Then:

1. Eigenvalues, eigenprojections and eigennilpotents for $h(\cdot)$ are (branches of) analytic functions with at most algebraic singularities at some (but not necessarily all) exceptional points.
2. Critical values (called *branch points* in [K2]; cf. Remark A.8) of $\lambda_{\ell,j}(\cdot)$ and $\Pi_{\ell,j}(\cdot)$ coincide, including the branching order (given in Definition A.4) of the branch points corresponding to these critical values. In particular, $\lambda_{\ell,j}(\cdot)$ single-valued near an exceptional point $\kappa_0 \in X$ (i.e., $\lambda_{\ell,j}(\cdot)$ constitutes a 1-cycle at $\kappa_0$; see Appendix A.2) implies that $\Pi_{\ell,j}(\cdot)$ and $N_{\ell,j}(\cdot)$ are single-valued near $\kappa_0$.
3. The critical values for $\lambda_{\ell,j}(\cdot)$ and $\Pi_{\ell,j}(\cdot)$ may or may not be critical values for $N_{\ell,j}(\cdot)$.

**Theorem B.9 (Butler)** [K2, II §1.6 Theorem 1.9]

Assume the same notation and conditions as in Theorem B.8. Let $\kappa_0 \in X$. If $\kappa_0$ is a critical value of $\lambda_{\ell,j}(\cdot)$ of order $n_\ell - 1 \geq 1$ (and therefore a critical value of $\Pi_{\ell,j}(\cdot)$ of the same order), then $\kappa_0$ is a pole of $\Pi_{\ell,j}(\cdot)$. In other words: Then $\Pi_{\ell,j}(\cdot)$ has a (convergent) Laurent series expansion in powers of $(\kappa - \kappa_0)^{1/n_\ell}$, which necessarily contains negative powers. In particular, one has $\|\Pi_{\ell,j}(\kappa)\| \to \infty$ $(\kappa \to \kappa_0)$.

## B.2 The structure of eigenvalues of the analytic matrix family $h(\kappa)$ in more detail

**Definition B.10** Let $k_\ell \in \mathbb{N} \cup \{\infty\}$ and let $\lambda_{\ell,j}(\kappa)$ be as in (B.15).

$\lambda_{\ell,j}(\cdot)$ branches in generation $k_\ell$ $:\Leftrightarrow$

$$\lambda_{\ell,j}(\kappa) = a_0^{(\ell,j)} + a_{n_\ell}^{(\ell,j)}\,\kappa + a_{2n_\ell}^{(\ell,j)}\,\kappa^2 + \ldots + a_{k_\ell \cdot n_\ell}^{(\ell,j)}\,\kappa^{k_\ell} + \sum_{\nu=1}^{\infty} a_{k_\ell \cdot n_\ell+\nu}^{(\ell,j)}\,\kappa^{k_\ell+\nu/n_\ell} \qquad (\kappa \in D_\varepsilon(0))$$

with $a_{k_\ell \cdot n_\ell+\nu}^{(\ell,j)} \neq 0$ for at least one $\nu$, $1 \leq \nu < n_\ell$.

Note that there is no branching of $\lambda_{\ell,j}(\kappa)$ $(\kappa \in D_\varepsilon(0))$, if and only if $k_\ell = \infty$. We remark that a branching in each generation $k_\ell \geq 1$ is possible. We introduce the $(k_\ell)$-jet

$$J_{\ell,j}^{(k_\ell)}(\kappa) := a_0^{(\ell,j)} + a_{n_\ell}^{(\ell,j)}\kappa + a_{2n_\ell}^{(\ell,j)}\kappa^2 + \ldots + a_{k_\ell \cdot n_\ell}^{(\ell,j)}\,\kappa^{k_\ell} \qquad (\kappa \in D_\varepsilon(0)) \tag{B.25}$$

of $\lambda_{\ell,j}(\kappa)$ for some $k_\ell \geq 1$. This is the part of $\lambda_{\ell,j}(\kappa)$, where no fractional exponents arise. And we define the $(k_\ell,\infty)$-jet of $\lambda_{\ell,j}(\kappa)$ (the part which contains fractional exponents):

$$J_{\ell,j}^{(k_\ell,\infty)}(\kappa) := \sum_{\nu=1}^{\infty} a_{k_\ell \cdot n_\ell+\nu}^{(\ell,j)}\,\kappa^{k_\ell+\nu/n_\ell} \qquad (\kappa \in D_\varepsilon(0)) \tag{B.26}$$

with $a_{k_\ell \cdot n_\ell+\nu}^{(\ell,j)} \neq 0$ for at least one $\nu$, $1 \leq \nu < n_\ell$.

**Lemma B.11** Assume that $\lambda_{\ell,j}(\cdot)$, given by (B.15), branches in generation $k_\ell$. Then

$$J_{\ell,j}^{(k_\ell)}(\kappa) = J_{\ell,j'}^{(k_\ell)}(\kappa) =: J_\ell^{(k_\ell)}(\kappa) \qquad (\kappa \in U_\varepsilon(0),\ j, j' \in \{1,\ldots,n_\ell\})\,. \tag{B.27}$$

**Proof**: Under analytic continuation around $\kappa = 0$ the branches $\lambda_{\ell,j}(\cdot)$ $(j \in \{1,\ldots,n_\ell\})$ transform one into another according to Remark B.4. But $J_{\ell,j}^{(k_\ell)}(\kappa)$ $(\kappa \in D_\varepsilon(0),\ j \in \{1,\ldots,n_\ell\})$ is a polynomial in $\kappa$ of degree $k_\ell$, and thus an analytic function of $\kappa$. Thus it does not change under analytic continuation.

■

By combining (B.15), (B.25), (B.26) and (B.27), each eigenvalue $\lambda_{\ell,j}(\cdot)$, $\ell \in \{1,\ldots,r\}$, $j \in \{1,\ldots,n_\ell\}$, has the representation

$$\lambda_{\ell,j}(\kappa) = J_\ell^{(k_\ell)}(\kappa) + J_{\ell,j}^{(k_\ell,\infty)}(\kappa) \tag{B.28}$$

for all $\kappa \in D_\varepsilon(0)$ and some $k_\ell \geq 1$. This motivates the following

**Definition B.12** We call $m(J_\ell^{(k_\ell)}) := n_\ell$ the multiplicity of the jet $J_\ell^{(k_\ell)}$.

**Remark B.13** Recall that $n_\ell$ has been defined as the degree of $\chi_\ell(\kappa,\lambda)$ in the unique factorization (B.14). The number $n_\ell$ coincides with the number of sheets (introduced in Appendix A.1.2) for the algebaric function corresponding to $\chi_\ell(\kappa,\lambda)$; see in particular Remark A.14 (Remark A.15, respectively).

For further reference we note

**Lemma B.14** Assume (B.9), (B.13), (B.28). Let $\ell \in \{1,\ldots,r\}$. Then the following statements are equivalent:

(1) $\lambda_{\ell,j} : D_\varepsilon(0) \to \mathbb{C}$ extends to an analytic function $\lambda_{\ell,j} : U_\varepsilon(0) \to \mathbb{C}$.

(2) There is no branching of $\lambda_{\ell,j}$.

(3) $\lambda_{\ell,j}$ forms a 1-cycle.

(4) $k_\ell = \infty$.

(5) $m(J_\ell^{(k_\ell)}) \stackrel{\text{Def.}}{=} n_\ell = 1$.

**Proof**:
(5)⇔(3): By Definition B.12, Appendix A.2 and Remark A.16.
(1)⇔(4)⇔(2): By Definition B.10.
(5)⇒(4): This is obvious by inserting $n_\ell = 1$ into Definition B.10.
(1)⇒(3): Let $\lambda_{\ell,j} : U_\varepsilon(0) \to \mathbb{C}$ be analytic. In particular, $\lambda_{\ell,j}$ is a single-valued analytic function. It thus forms a 1-cycle; cf. Remark B.4.

■

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